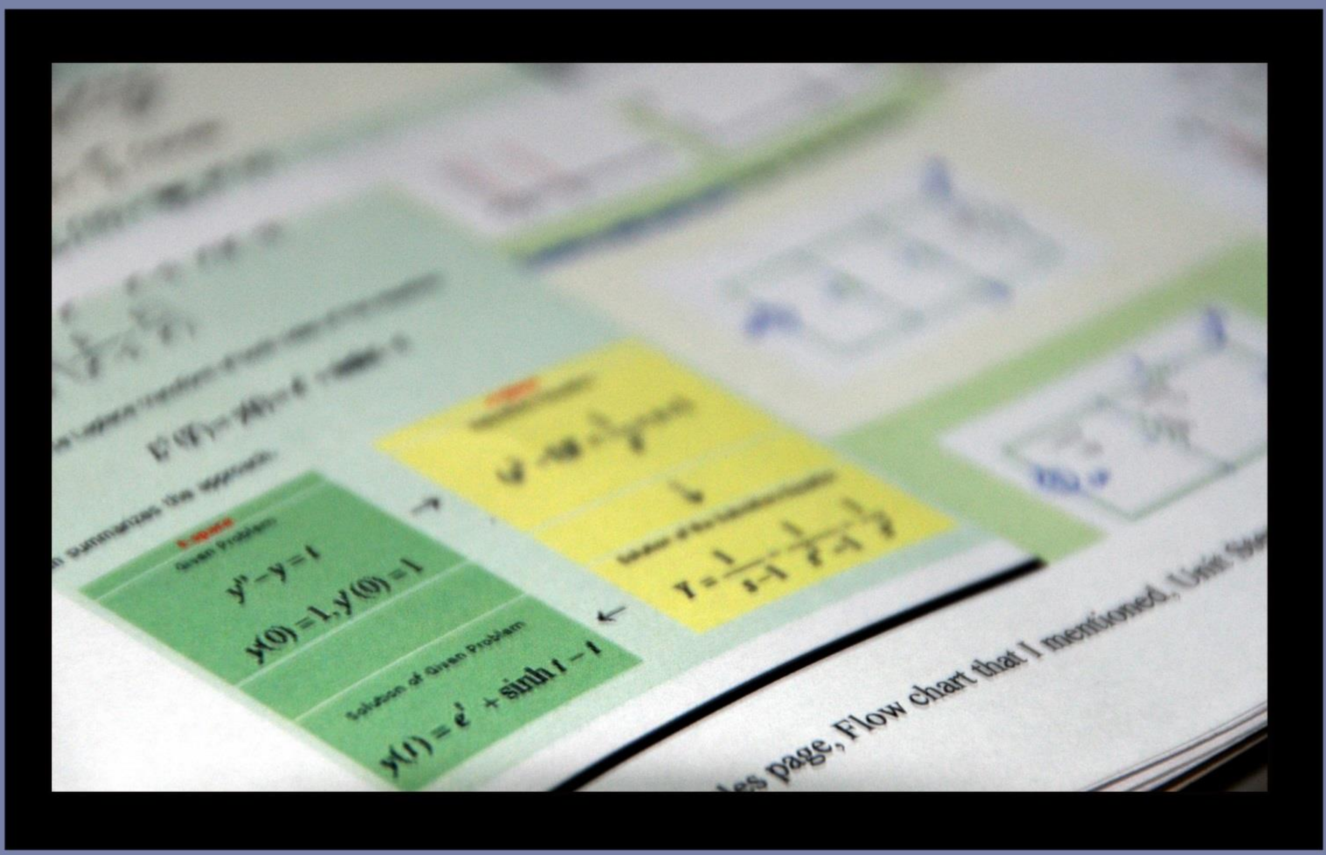
$$F(s) = \mathcal{L}\{f(t)\} = \int_{-\infty}^{+\infty} e^{-st} f(t) dt$$

Laplace Transforms

And Their Applications In Electric Circuits



Yas Ka

Content

Section 2

Section 3

Preface

Mathematics is so complicated that we should be crystal clear aware of the way we approach it. If we use mathematics as a tool, we shouldn't disappear among the equations by forgetting our aim. And if we want to make mathematics, we shouldn't be anxious weather it will be useful or not.

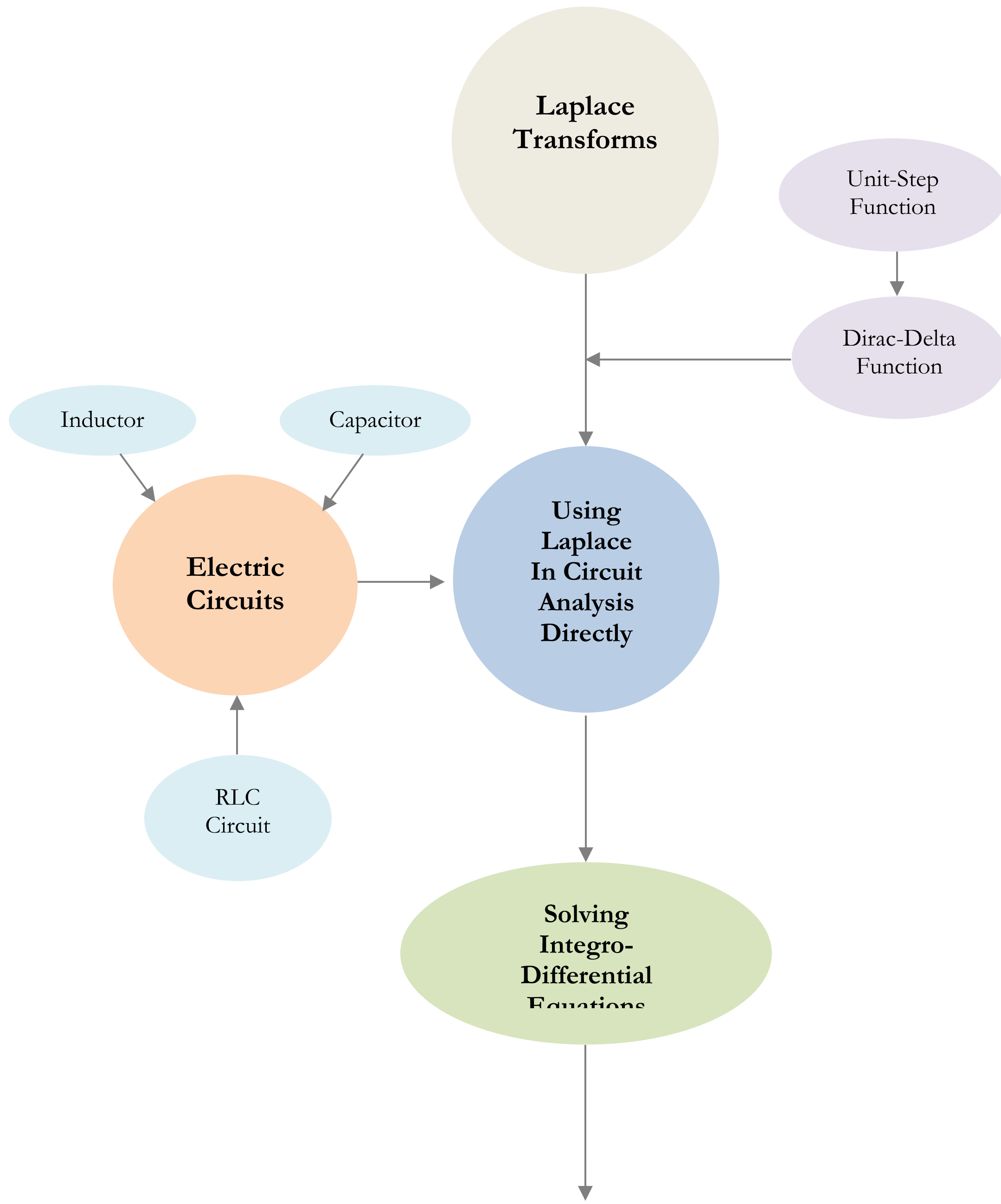
After investigating Laplace transforms and its applications, especially applications in electric circuit analysis, it is understood that there are tree steps in the development of the subject.

The first one is purely mathematical approach which we are deeply interested in definitions, theorems, and proofs.

The second step is a kind of transient step which easily lead us to third one. In this step, electric circuit analysis forces us to look for a mathematical tool to solve integro-differential equations which explain the behavior of the circuits.

And in the third step, we develop an s- domain based circuit analysis concept, seeing the fact that we are doing the same things during the solution of integro-differential equations. After this development, forgetting the theory of Laplace transforms will not cause us to fail in reaching to a solution in our electrical analysis problem.

This book is based on an undergraduate work.


Laplace Transforms
Unit-Step Function
Dirac-Delta Function
Inductor
Capacitor
Electric Circuits
Using Laplace In Circuit Analysis Directly
RLC Circuit
Solving Integro-Differential Equations

Biography of Pierre-Simon Laplace

Pierre-Simon Laplace's father, Pierre Laplace, was comfortably well off in the cider trade. Laplace's mother, Marie-Anne Sochon, came from a fairly prosperous farming family who owned land at Tourgéville. Many accounts of Laplace say his family were 'poor farming people' or 'peasant farmers' but these seem to be rather inaccurate although there is little evidence of academic achievement except for an uncle who is thought to have been a secondary school teacher of mathematics. This is stated in [1] in these terms:-

> *There is little record of intellectual distinction in the family beyond what was to be expected of the cultivated provincial bourgeoisie and the minor gentry.*

Laplace attended a Benedictine priory school in Beaumont-en-Auge, as a day pupil, between the ages of 7 and 16. His father expected him to make a career in the Church and indeed either the Church or the army were the usual destinations of pupils at the priory school. At the age of 16 Laplace entered Caen University. As he was still intending to enter the Church, he enrolled to study theology. However, during his two years at the University of Caen, Laplace discovered his mathematical talents and his love of the subject. Credit for this must go largely to two teachers of mathematics at Caen, C Gadbled and P Le Canu of whom little is known except that they realised Laplace's great mathematical potential.

Once he knew that mathematics was to be his subject, Laplace left Caen without taking his degree, and went to Paris. He took with him a letter of introduction to d'Alembert from Le Canu, his teacher at Caen. Although Laplace was only 19 years old when he arrived in Paris he quickly impressed d'Alembert. Not only did d'Alembert begin to direct Laplace's mathematical studies, he also tried to find him a position to earn enough money to support himself in Paris. Finding a position for such a talented young man did not prove hard, and Laplace was soon appointed as professor of mathematics at the Ecole Militaire.

Historical Summary of Laplace Transforms

Integral Transforms date back to the work of Leonard Euler (1763 and 1769), who considered them essentially in the form of the inverse Laplace transform in solving second-order, linear differential equations. Even Laplace in his great work, *Theorie analytique des probabilites* (1812), credits Euler with introducing integral transforms. It is Spitzer (1878) who attached the name of Laplace to the expression

$$y = \int_a^b e^{sx} \Phi(s) ds$$

employed by Euler. In this form it is substituted into the differential equation where y is the unknown function of the variable x.

In the late 19th century, the Laplace transform was extended to its complex form by Poincare and Picherle, rediscovered by Petzval, and extended to two variables by Picard, with further investigations conducted by Abel and many others.

The first application of the modern Laplace transform occurs in the work of Bateman(1910), who transforms equations arising from Rutherford's work on radioactive decay

$$\frac{dP}{dt} = -\lambda_i P$$

by setting

$$p(x) = \int_0^\infty e^{-xt} P(t) dt$$

and obtaining the transformed equation. Bernstein (1920) used the expression

$$f(s) = \int_0^\infty e^{-su} \Phi(u) du$$

calling it the Laplace transformation, in his work on theta functions. The modern approach was given particular impetus by Doetsch in the 1920s and 30s; he applied the Laplace transform to differential, integral and integro-differential equations. This body of work culminated in his foundational 1937 text, *Theorie und Anwendungen der Laplace Transformation.*

No account of the Laplace transformation would be complete without mention of the work of Oliver Heaviside, who produced (mainly in the context of electrical engineering) a vast body of what is termed the "operational calculus". This material is scattered throughout his three volumes, Electromagnetic Theory (1894, 1899, 1912), and bears many similarities to the Laplace transform method. Although Heaviside's calculus was not entirely rigorous, it did find favor with electrical engineers as a useful technique for solving their problems. Considerable research went into trying the make the Heaviside calculus rigouros and connecting it with the Laplace transform. One such effort was that of Bromwich, who, among others, discovered the inverse transform

$$X(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{is} x(s)ds$$

for *y* lying to the right of all the singularities of the function s.

Basic Idea About Laplace Transforms

The Laplace transform is a wonderful tool for solving ordinary and partial differential equations and has enjoyed much success in this realm.

Ordinary and partial differential equations describe the way certain quantities vary with time, such as the current in an electrical circuit, the oscillations of a vibrating membrane, or the flow of heat through an insulated conductor. These equations are generally coupled with initial conditions that describe the state of the system at time t=0.

Laplace transforms literally transform the original differential equation into an elementary algebraic expression. This algebraic equation can then simply be transformed once again, into the solution of the original problem. This technique is known as the Laplace transform method.

The process of solution consists of three main steps:

1) The given 'hard' problem is transformed into a 'simple' equation which is called subsidiary equation.

2) The subsidiary equation is solved by purely algebraic manipulations.

3) The solution of subsidiary equation is transformed back to obtain the solution of the given problem.

In this way Laplace transforms reduce the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role similar to that of integral tables in integration.

The switching operation of calculus to algebraic operations on transforms is called operational calculus, a very important area of applied mathematics, and the Laplace transform method is practically the most important method for this purpose.

Definition

Let $f(t)$ be a given function that is defined for all e^{-st} .We multiply $f(t)$ by e^{-st} and integrate with respect to *t* from zero to infinity.Then, if the resulting integral exists (that is, has some finite value), it is a function of s, say, *F(s)*:

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

This function *F(s)* of the variable s is called the **Laplace Transform** of the original function *f(t)*, and will be denoted by ***L{ f }***.Thus,

$$F(s) = L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

Examples

Example 1:

$$L\{1\} = ?$$

$$L\{1\} = \int_0^\infty e^{-st}dt = \lim_{T\to\infty} -\frac{1}{s}e^{-st}\Big|_0^T = -\frac{1}{s}(e^{-sT} - 1) = \frac{1}{s}$$

Example 2:

$$f(t) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$$

Now let’s find the Laplace transform of this function.

$$L\{f(t)\} = \int_o^\infty e^{-st} f(t)dt$$

$$L\{f(t)\} = \int_o^a e^{-st} f(t)dt + \int_a^\infty e^{-st} f(t)dt$$

$$L\{f(t)\} = \int_o^a e^{-st}.0dt + \int_a^\infty e^{-st}.1dt$$

$$L\{f(t)\} = \int_a^\infty e^{-st}dt = -\frac{e^{-st}}{s}\Big|_a^\infty = -\frac{1}{s}\left(e^{-s\infty} - e^{-sa}\right)$$

$$L\{f(t)\} = \frac{e^{-as}}{s}$$

Example 3:

$$L\{\cosh at\} = ?$$

$$L\{\cosh at\} = \int_0^\infty e^{-st} \cosh at dt$$

$$\cosh at = \frac{e^{at} + e^{-at}}{2}$$

$$= \int_0^\infty e^{-st} \left(\frac{e^{at} + e^{-at}}{2}\right) dt$$

$$= \frac{1}{2}\left(\int_0^\infty e^{(-s+a)t} dt + \int_0^\infty e^{(-s-a)t} dt\right)$$

$$= \lim_{T \to \infty} \frac{1}{2}\left(\frac{1}{a-s} e^{(a-s)t}\Big|_0^T + \frac{1}{-s-a} e^{(-s-a)t}\Big|_0^T\right)$$

$$= \frac{1}{2}\left(0 - \frac{1}{a-s} + 0 + \frac{1}{-s-a}\right)$$

$$= \frac{1}{2}\left(\frac{1}{s-a} + \frac{1}{s+a}\right) = \frac{s}{s^2 - a^2}$$

Example 4:

$$L\{\cos at\} = ?$$

$$L\{\cos at\} == \int_0^\infty e^{-st} \cos at dt$$

So, we have to use integration by parts. Then,

$$e^{-st} = u$$
$$-se^{-st} = du$$
$$\cos atdt = dv$$
$$\frac{1}{a}\sin at = v$$
$$\int udv = uv - \int vdu$$

$$\int_0^\infty e^{-st}\cos atdt = \lim_{T\to\infty}\frac{e^{-st}\sin at}{a}\Big|_0^T + \frac{s}{a}\int_0^\infty e^{-st}\sin atdt$$

It can be easily seen that

$$\lim_{T\to\infty}\frac{e^{-st}\sin at}{a}\Big|_0^T = 0$$

Then we have

$$\int_0^\infty e^{-st}\cos atdt = \int_0^\infty \frac{s}{a}e^{-st}\sin atdt$$

We have to use integration by parts again.

$$e^{-st} = u$$
$$-se^{-st} = du$$
$$\sin atdt = dv$$
$$-\frac{1}{a}\cos at = v$$
$$\int udv = uv - \int vdu \Rightarrow$$
$$\int_0^\infty e^{-st}\cos atdt = \frac{s}{a}\left(\frac{-e^{-st}\cos at}{a} - \frac{s}{a}\int_0^\infty e^{-st}\cos atdt\right)$$

If we call

$$\int_0^\infty e^{-st}\cos atdt = A$$

then,

$$A = -\frac{se^{-st}\cos at}{a^2}\Big|_0^T - \frac{s^2}{a^2}A = \frac{s}{a^2} - \frac{s^2}{a^2}A \Rightarrow$$

$$A = \frac{s}{s^2 + a^2}$$

So, the result is

$$L\{\cos at\} = \int_0^\infty e^{-st}\cos at dt = \frac{s}{s^2 + a^2}$$

Example 5:

$$f(t) = \begin{cases} \sin \omega t, & 0 < t < \dfrac{\pi}{\omega} \\ 0, & \dfrac{\pi}{\omega} < t \end{cases}$$

$$L(f(t) = \int_0^{\pi/\omega} e^{-st}\sin \omega t dt + e^{-st}.0dt \int_{\pi/\omega}^{\infty} e^{-st}.0dt =$$

$$= \int_0^{\pi/\omega} e^{-st}\sin \omega t dt$$

Using integration by parts,

$$\int u dv = uv - \int v du$$

$$\sin \omega t = u$$

$$\omega \cos \omega t dt = du$$

$$e^{-st}dt = dv$$

$$-\frac{1}{s}e^{-st} = v$$

$$\Rightarrow \int_0^{\pi/\omega} e^{-st}\sin \omega t dt = -\frac{e^{-st}\sin \omega}{s} + \int_o^{\pi/\omega} \frac{\omega}{s} e^{-st}\cos \omega t dt$$

So, we have again an integral to solve with integration by parts.

$$\int e^{-st}\cos\omega t dt = -\frac{e^{-st}\cos\omega t}{s} - \frac{\omega}{s}\int e^{-st}\sin\omega t dt$$

If we write what we have as a hole now.

$$\int e^{-st}\sin\omega t dt = -\frac{e^{-st}\sin\omega}{s} + \frac{\omega}{s}\left(-\frac{e^{-st}\cos\omega t}{s} - \frac{\omega}{s}\int e^{-st}\sin\omega t dt\right)$$

$$\text{let} \quad \int e^{-st}\sin\omega t dt = A \quad \text{then}$$

$$A = -\frac{e^{-st}\sin\omega t}{s} + \frac{\omega e^{-st}\cos\omega t}{s^2} - \frac{\omega^2}{s^2}A$$

$$A(1+\frac{\omega^2}{s^2}) = -\frac{e^{-st}\sin\omega t}{s} + \frac{\omega e^{-st}\cos\omega t}{s^2}$$

$$A = \frac{-1}{(s^2+\omega^2)}\left[e^{-st}\left(s\sin\omega t + \omega\cos\omega t\right)\right]$$

So now, we can calculate

$$\int_0^{\pi/\omega} e^{-st}\sin\omega t dt = \frac{-1}{(s^2+\omega^2)}\left[e^{-st}\left(s\sin\omega t + \omega\cos\omega t\right)\right]\Big|_0^{\frac{\pi}{\omega}}$$

$$= \frac{-1}{(s^2+\omega^2)}\left[e^{\frac{-s\pi}{\omega}}(s\sin\frac{\omega\pi}{\omega} + \omega\cos\frac{\omega\pi}{\omega}) - e^0(s\sin 0 + \omega\cos 0)\right]$$

$$= \frac{-1}{(s^2+\omega^2)}\left[e^{\frac{-s\pi}{\omega}}(0-\omega) - \omega\right] = \frac{-1}{(s^2+\omega^2)}\left(-\omega e^{\frac{-s\pi}{\omega}} - \omega\right)$$

And finally we can say

$$L\{f(t)\} = \int_0^{\pi/\omega} e^{-st}\sin\omega t dt = \frac{\omega}{s^2+\omega^2}\left(1+e^{\frac{-s\pi}{\omega}}\right)$$

Existence Theorem

Let f(t) be a function that is piecewise continues on every finite interval on the range of $t \geq 0$ and satisfies

$$| f(t) | \leq Me^{\gamma t}$$

For all $t \geq 0$ and for some constants γ and *M*. Then the Laplace transform of *f(t)* exists for all $s \geq \gamma$

Properties

Linearity:

$$L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$$

Examples:

1)

$$L\{\sin 4t - 3\cos 7t\} = L\{\sin 4t\} - L\{3\cos 7t\}$$
$$= \frac{4}{s^2 + 16} - \frac{3s}{s^2 + 49}$$

2)

$$L\{\sinh 3t + 3\cosh 5t\} = L\{\sinh 3t\} + L\{3\cos 5t\}$$
$$= \frac{4}{s^2 - 16} + \frac{3s}{s^2 - 25}$$

3)

$$L\{\sin^2 3t + 2\cos^2 4t\} = ?$$
$$\sin^2 x = \frac{1-\cos 2x}{2}$$
$$\cos^2 x = \frac{1+\cos 2x}{2} \Rightarrow$$
$$L\left\{\frac{1-\cos 6t}{2}\right\} + L\left\{\frac{2(1+\cos 8t)}{2}\right\}$$
$$= \frac{1}{2}\left[L\{1\} - L\{\cos 6t\}\right] + L\{1\} + L\{\cos 8t\}$$
$$= \frac{1}{2}\left[\frac{1}{s} - \frac{s}{s^2+36}\right] + \frac{1}{s} + \frac{s}{s^2+64}$$

4)

$$L\{e^{-2t}\sinh 4t + e^{3t}t^2\} = ?$$
$$= L\{e^{-2t}\sinh 4t\} + L\{e^{3t}t^2\}$$
$$= \frac{4}{(s+2)-16} + \frac{2}{(s-3)^3}$$

Shifting Property:

$$L\{e^{at}f(t)\} = F(s-a)$$

Examples:

1)

$$L\{e^{4t}\sin 3t\} = \frac{3}{(s-4)^2+9}$$

2)

$$L\{e^{2t}t^2\} = ?$$
$$L\{t^2\} = \frac{2}{s^3}$$
$$L\{e^{2t}t^2\} = \frac{2}{(s-2)^3}$$

3)

$$L\{e^{4t}\sin 3t\} = ?$$
$$L\{\sin 3t\} = \frac{3}{s^2+9} \Rightarrow$$
$$L\{e^{4t}\sin 3t\} = \frac{3}{(s-4)^2+9}$$

4)

$$L\{e^{-\alpha t}(A\cos(\beta t) + B\sin(\beta t)) = ?$$
$$= A\frac{s+\alpha}{(s+\alpha)^2+\beta^2} + B\frac{\beta}{(s+\alpha)^2+\beta^2}$$
$$= \frac{A(s+\alpha)}{(s+\alpha)^2+\beta^2}$$

5)

$$L\{e^{-t}(a_0 + a_1 t + \ldots + a_n t^n)\} = ?$$
$$L\{e^{-t}(a_0 + a_1 t + \ldots + a_n t^n)\} = L\{e^{-t}a_0 + e^{-t}a_1 t + \ldots + e^{-t}a_n t^n)\}$$
$$= \frac{a_0}{s+1} + \frac{a_0}{(s+1)^2} + \ldots + \frac{a_n n!}{(s+1)^{n+1}}$$

Change of scale:

$$L\{\sin 3t\} = ?$$

Since we know

$$L\{\sin t\} = \frac{1}{s^2+1}$$

Using the formula

$$L\{f(at)\} = \frac{1}{a}F\left(\frac{s}{a}\right)$$

we can say

$$L\{\sin 3t\} = \frac{1}{3}\frac{1}{\left(\frac{s}{3}\right)^2+1} = \frac{3}{s^2+9}$$

Laplace of Derivative

Now, we discuss and apply the most crucial property of Laplace transform. Roughly speaking, differentiation of function corresponds to the multiplication of transforms by *s*, and integration of functions corresponds to the division of transforms by *s*. Hence, the Laplace transform replaces operations of calculus by operations of algebra on transforms. This is Laplace's basic idea, for which we should admire him.

Laplace transform of the derivative of *f(t)*

Suppose that *f(t)* is continuous for all t>=0, satisfies existence theorem, for some γ and M, and has a derivative *f′(t)* that is piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of the derivative *f′(t)* exists when s>γ, and

$$L\{f'\} = sL\{f\} - f(0)$$

Laplace transform of the derivative of any order *n*

Let *f(t)* and its derivatives *f′(t), f″ (t), …,* $f^{(n-1)}(t)$ be continuous functions for all t>=0, satisfying existence theorem, for some γ and M, and let the derivative $f^{(n)}(t)$ be piecewise continuous on every finite interval in the range $t \geq 0$. Then the Laplace transform of $f^{(n)}(t)$ exists when s>γ, and is given by

$$L\{f^{n}(t)\} = s^{n}F(s) - s^{n-1}f(0) - s^{n-2}f^{'}(0) - ... - s^{2}f^{n-3}(0) - s^{1}f^{n-2}(0) - f^{n-1}(0)$$

Examples

1)

$$L\{(e^{4t})'\} = ?$$
$$L\{f'(t)\} = sF(s) - f(0)$$
$$L\{(e^{4t})'\} = \frac{s}{s-4} - e^{0}$$
$$= \frac{s}{s-4} - 1$$

2)

$$L\{(\cos 4t)''\} = ?$$
$$L\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$$
$$L\{(\cos 4t)''\} = s^2\frac{s}{s^2+16} - s\cos 0 + 4\sin 0 = \frac{s^3}{s^2+16} - s$$

3)
Find the Laplace transform of $f(t) = \cos\omega t$ using derivative formulas.

$$L\{\cos\omega t\} = ?$$
$$(\cos\omega t)' = -\sin\omega t$$
$$(\cos\omega t)'' = -\omega^2\cos\omega t$$

so

$$L\{(\cos\omega t)'\} = L\{-\omega^2\cos\omega t\} = s^2L\{\cos\omega t\} - s$$
$$-\omega^2L\{\cos\omega t\} = s^2L\{\cos\omega t\} - s$$
$$L\{\cos\omega t\} = \frac{s}{s^2+\omega^2}$$

Laplace of Integral

Since differentiation and integration are inverse processes, and since, roughly speaking, differentiation of a function corresponds to the multiplication of its transform by s, we expect integration of a function to correspond to the division of its transform by s, because division is the inverse operation of multiplication:

Integration of *f(t)*

If *f(t)* is piecewise continuous and satisfies an inequality of the form $|f(t)| \le Me^{\gamma t}$ (Existence of Laplace Transforms) for some γ and *M*

$$L\left\{\int_o^t f(\tau)d\tau\right\} = \frac{L\{f(t)\}}{s} = \frac{F(s)}{s} \quad s > 0,\ s > \gamma$$

This equation has a useful companion, which we obtain by writing

$$L\{f(t)\} = F(s)$$

,interchanging the two sides and taking the inverse transform on both sides. Then

$$L^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t f(\tau)d\tau$$

Example 1:

$$L\{f(t)\} = \frac{1}{s(s^2+\omega^2)} \Rightarrow L^{-1}\{f(t)\} = ?$$

If we recall that

$$L^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\sin \omega t$$

From this and integration of Laplace we obtain the answer

$$L^{-1}\left\{\frac{1}{s^2+\omega^2}\right\} = \frac{1}{\omega}\int_o^t \sin \omega\tau d\tau = \frac{1}{\omega^2}(1-\cos\omega t)$$

Example 2: If we expand the preceding example

$$L\{f(t)\} = \frac{1}{s^2(s^2+\omega^2)}.\ Find\ \ f(t).$$

Applying integration of Laplace formula to the result we obtained above, the solution is

$$L^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_o^t (1-\cos\omega\tau)d\tau = \frac{1}{\omega^2}\left(t-\frac{\sin\omega t}{\omega}\right)$$

Multiplication by t

It can be shown that if f(t) satisfies the conditions of existence theorem the derivative of its transform with respect to s can be obtained by differentiating under the integral sign with respect to s.

$$F'(s) = -\int_0^\infty e^{-st}[tf(t)dt$$

Consequently,

$$L\{tf(t)\} = -F'(s)$$

Differentiation of the transform of a function corresponds to the multiplication of the function by $-t$.

Generally we can say

$$L\{t^n f(t)\} = (-1)^n \frac{d^nF(s)}{ds^n}$$

Examples:

1)

$$L\{t^2 \sin 2t\} = \frac{-4}{(s^2+4)^2} + \frac{16s^2}{(s^2+4)^3}$$

2)

$$L\{e^{2t} t \cosh 3t\} = \frac{(s-2)^2+9}{((s-2)^2-9)^2}$$

3)

$$L\{t^2 \sin 2t\} = ?$$

$$L\{t^n f(t)\} = (-1)^n F^n(s) = (-1)^n \frac{d^n F(s)}{ds^n}$$

$$L\{\sin 2t\} = \frac{2}{s^2+4}$$

$$L\{t^2 \sin 2t\} = (-1)^2 \frac{d^2(\frac{2}{s^2+4})}{ds^2}$$

$$= \frac{d}{ds}(2(-1)(s^2+4)^{-2}2s) = -4s(s^2+4)^{-2}$$

$$= -4(s^2+4)^{-2} + 16s^2(s^2+4)^{-3}$$

$$= \frac{-4}{(s^2+4)^2} + \frac{16s^2}{(s^2+4)^3}$$

4)

$$L\{e^{2t} t \cosh 3t\} = ?$$

$$L\{t \cosh 3t\} = \frac{s}{s^2-9}$$

$$= (-1)'\left(\frac{s}{s^2-9}\right)'$$

$$= (-1)\left(\frac{(s^2-9)-2s^2}{(s^2-9)^2}\right)$$

$$= \frac{(s^2+9)}{(s^2-9)^2}$$

Now, using shifting property

$$L\{e^{2t} t \cosh 3t\} = \frac{(s-2)^2+9}{((s-2)^2-9)^2}$$

Another approach to this problem can be using shifting property first.

$$L\{\cosh t\} = \frac{s}{s^2 - 9}$$

$$L\{e^{2t} \cosh t\} = \frac{s-2}{(s-2)^2 - 9}$$

$$= \frac{s-2}{s^2 - 4s - 5}$$

Now, using the formula

$$L\{t^n f(t)\} = (-1)^n F^n(s) = (-1)^n \frac{d^n F(s)}{ds^n}$$

We have

$$= (-1)\left(\frac{s-2}{s^2 - 4s - 5}\right)'$$

$$= (-1)\left(\frac{s^2 - 4s - 5 - (s-2)(2s-4)}{(s^2 - 4s - 5)^2}\right)$$

$$= (-1)\frac{s^2 - 4s - 5 - (2s^2 - 4s - 4s + 8)}{(s^2 - 4s - 5)^2}$$

$$= \frac{-s^2 + 4s + 5 + 2s^2 - 4s - 4s + 8}{(s^2 - 4s - 5)^2}$$

$$= \frac{s^2 - 4s + 13}{(s^2 - 4s - 5)^2} = \frac{(s-2)^2 + 9}{((s-2)^2 - 9)^2}$$

Integration of Transforms (Division by t)

Similarly, if *f(t)* satisfies the conditions of the existence theorem and the limit of *f(t)/t*. as *t* approaches 0 from the right, exists, then

$$L\{\frac{f(t)}{t}\} = \int_s^{\infty} F(u)du \qquad (s > \gamma) \tag{1}$$

in this manner, integration of the transform of a function *f(t)* corresponds to the division of *f(t)* by *t*. Equivalently,

$$L\{\int_s^{\infty} F(u)du\} = \frac{f(t)}{t}.$$

In fact, from the definition it follows that

$$\int_s^{\infty} F(u)du = \int_s^{\infty}\left[\int_0^{\infty} e^{-ut} f(t)dt\right]du,$$

and it can be shown that under the above assumption we may reverse the order of integration, that is,

$$\int_s^{\infty} F(u)du = \int_s^{\infty}\left[\int_0^{\infty} e^{ut} f(t)du\right]dt = \int_0^{\infty} f(t)\left[\int_s^{\infty} e^{-utdu}\right]dt.$$

The integral over u on the right equals e^{-ut}/t when $(s > \gamma)$, and, therefore,

$$\int_s^{\infty} F(u)du = \int_0^{\infty} e^{ut}\frac{f(t)}{t}dt = L\left\{\frac{f(t)}{t}\right\} \qquad s > \gamma$$

Example 2: Integration of transforms

Find the inverse transform of the function $\ln\left(1+\frac{\omega^2}{s^2}\right)$

Solution. By differentiation,

$$-\frac{d}{ds}In\left(1+\frac{\omega^2}{s^2}\right) = \frac{2\omega^2}{s(s^2+\omega^2)} = \frac{2}{s} - 2\frac{s}{s^2+\omega^2},$$

where the last equality can be readily verified by direct calculation. This is our present *F(s)*. It is the derivative of the given function (times –1), so that the later is the integral of *F(s)* from s to ∞. We obtain

$$f(t) = L^1(F) = L^{-1}\left\{\frac{2}{s} - 2\frac{s}{s^2+\omega^2}\right\} = 2 - 2\cos\omega t$$

This function satisfies the conditions under which (1) holds. Therefore,

$$L^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\} = L^{-1}\left\{\int_s^{\infty} F(u)du\right\} = \frac{f(t)}{t}$$

Our result is

$$L^{-1}\left\{\ln\left(1+\frac{\omega^2}{s^2}\right)\right\} = \frac{2}{t}(1-\cos\omega t)$$

Inverse Laplace Transforms

Usage of Table

In fact, we can say usage of table to find the inverse Laplace of a function means guessing whose Laplace is that function. Inverse Laplace transforms of lots of functions exist on tables. After making some manipulations you get the forms in tables and then you can directly see the solution.

Examples:

1)

$$L^{-1}\left\{\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}\right\}=L^{-1}\left\{\frac{5}{s^2}+\frac{4}{s^3}-\frac{2s}{s^2+9}-\frac{18}{s^2+9}\right\}$$

$$L^{-1}\left\{\frac{5s+4}{s^3}-\frac{2s-18}{s^2+9}\right\}=5t+2t^2-2\cos 3t-6\sin 3t$$

2)

$$L^{-1}\left\{\frac{6}{2s-3}\right\}=L^{-1}\left\{\frac{6}{2\left(s-\frac{3}{2}\right)}\right\}=3L^{-1}\left\{\frac{1}{\left(s-\frac{3}{2}\right)}\right\}=3e^{\frac{3}{2}t}$$

3)

$$L^{-1}\left\{\frac{3+4s}{9s^2+16}\right\}=L^{-1}\left\{\frac{3}{9s^2+16}+\frac{4s}{9s^2+16}\right\}=L^{-1}\left\{\frac{3}{9\left(s^2+\frac{16}{9}\right)}+\frac{4s}{9\left(s^2+\frac{16}{9}\right)}\right\}$$

$$=L^{-1}\left\{\frac{3\left(\frac{4}{3}\right)}{9\left(\frac{4}{3}\right)\left[s^2+\left(\frac{4}{3}\right)^2\right]}+\frac{4s}{9\left(s^2+\frac{16}{9}\right)}\right\}=\frac{1}{4}L^{-1}\left\{\frac{\frac{4}{3}}{\left[s^2+\left(\frac{4}{3}\right)^2\right]}\right\}+\frac{4}{9}L^{-1}\left\{\frac{4s}{9\left(s^2+\frac{16}{9}\right)}\right\}$$

$$L^{-1}\left\{\frac{3+4s}{9s^2+16}\right\}=\frac{1}{4}\sin\frac{4}{3}t+\frac{4}{9}\cos\frac{4}{3}t$$

4)

$$L^{-1}\left\{\frac{6s-4}{s^2-4s+20}\right\}=L^{-1}\left\{\frac{6s-4}{s^2-4s+4+16}\right\}=L^{-1}\left\{\frac{6(s-2)+12-4}{(s-2)^2+16}\right\}$$

$$=L^{-1}\left\{\frac{6(s-2)}{(s-2)^2+16}\right\}+L^{-1}\left\{\frac{8}{(s-2)^2+16}\right\}$$

$$=6e^{2t}\cos 4t+2e^{2t}\sin 4t$$

5)

$$L^{-1}\left\{\frac{4s+12}{s^2+8s+16}\right\}=L^{-1}\left\{\frac{4s+12}{(s+4)^2}\right\}=L^{-1}\left\{\frac{4s+12+4-4}{(s+4)^2}\right\}$$

$$=L^{-1}\left\{\frac{4(s+4)-4}{(s+4)^2}\right\}=4e^{-4t}-4e^{-4t}t$$

Partial Fractions Method

Partial Fractions are needed to obtain the solution $y(t)=L^{-1}\{f(t)\}$ of a problem from the solution *Y(s)* of the subsidiary equation (we will talk about later) because *Y* usually comes out as a quotient of two polynomials.,

$$Y(s)=\frac{F(s)}{G(s)}$$

and the inverse of a partial fraction *P* is easy to get from table and the first shifting theorem. Now, we will solve two simple examples. In fact, partial fractions is a very detailed subject.

1)

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\}=?$$

$$\frac{s+3}{(s+1)(s-3)}=\frac{A}{s+1}+\frac{B}{s-3}$$

$$A(s-3)+B(s+1)=s+3$$

$$s = 3 \Rightarrow B = \frac{3}{2}$$

$$s = -1 \Rightarrow A = -\frac{1}{2}$$

then

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\} = L^{-1}\left\{\frac{-1}{2(s+1)} + \frac{3}{2(s-3)}\right\}$$

$$= -\frac{1}{2}e^{-t} + \frac{3}{2}e^{3t}$$

Here we can use a practical method.

We can find A directly by omitting (s+1) in

$$\frac{s+3}{(s+1)(s-3)}$$

and then for s= -1, which makes s+1 zero

$$\frac{s+3}{s-3} = \frac{-1+3}{-1-3} = -\frac{1}{2} = A$$

And for B, omitting s-3 in

$$\frac{s+3}{(s+1)(s-3)}$$

and setting s=3, which makes s-3 zero we have

$$\frac{s+3}{s+1} = \frac{3+3}{3+1} = \frac{3}{2} = B$$

2)

$$L^{-1}\left\{\frac{2s-1}{(s+1)(s^2+1)}\right\} = ?$$

$$\frac{2s-1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1}$$

$$A(s^2+1) + (Bs+C)(s+1) = 2s-1$$

$$s = -1 \Rightarrow A = -\frac{3}{2}$$

$$s = 0 \Rightarrow C = \frac{1}{2}$$

$$s = 1 \Rightarrow B = \frac{3}{2}$$

then

$$L^{-1}\left\{\frac{2s-1}{(s+1)(s^2+1)}\right\} = L^{-1}\left\{\frac{-3}{2(s+1)} + \frac{\frac{3}{2}s + \frac{1}{2}}{s^2+1}\right\}$$

$$= L^{-1}\left\{\frac{-3}{2(s+1)} + \frac{3}{2}\frac{s}{(s^2+1)} + \frac{1}{2(s^2+1)}\right\}$$

$$= -\frac{3}{2}e^{-t} + \frac{3}{2}\cos t + \frac{1}{2}\sin t$$

In most circuit analysis problems, four useful transform pairs will be enough to reach to the solution. This can be summarized in a table.

Nature of Roots	*F(s)*	$f(t)$
Distinct Real	$\frac{K}{s+a}$	$Ke^{-at}u(t)$
Repeated Real	$\frac{K}{(s+a)^2}$	$Kte^{-at}u(t)$
Distinct Complex	$\frac{K}{s+\alpha-j\beta} + \frac{K^*}{s+\alpha+j\beta}$	$2\|K\|e^{-\alpha t}\cos(\beta t+\theta)u(t)$
Repeated Complex	$\frac{K}{(s+\alpha-j\beta)^2} + \frac{K^*}{(s+\alpha+j\beta)^2}$	$2t\|K\|e^{-\alpha t}\cos(\beta t+\theta)u(t)$
Note: In pairs 1 and 2, K is a real quantity, whereas in pairs 3 and 4, is the complex quantity |K|∠θ.		

Residue Method

$$L^{-1}\{F(s)\} = f(t) = \sum \text{residue} \quad \text{of } e^{st}F(s) \text{ at poles } = R_1 + R_2 + \ldots + R_n$$

Simple Pole:

$$R_i = \lim_{s \to s_i} \left[(s - s_i)e^{st}F(s)\right]$$

Multiple Pole:

$$R_i = \lim_{s \to s_i} \frac{1}{(n-1)!} \frac{d}{ds^{n-1}} \left[(s - s_i)^n e^{st}F(s)\right]$$

Examples:

1)

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\} = ?$$

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\} = R_1 + R_2$$

$$R_1 = \lim_{s \to -1}\left[(s+1)\frac{(s+3)}{(s+1)(s-3)}e^{-t}\right] = -\frac{1}{2}e^{-t}$$

$$R_1 = \lim_{s \to 3}\left[(s-3)\frac{(s+3)}{(s+1)(s-3)}e^{3t}\right] = \frac{3}{2}e^{3t}$$

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\} = R_1 + R_2 = -\frac{1}{2}e^{-t} + \frac{3}{2}e^{3t}$$

2)

$$L^{-1}\left\{\frac{s+1}{s(s^2+s-6)}\right\} = ?$$

$$L^{-1}\left\{\frac{s+1}{s(s^2+s-6)}\right\} = R_1 + R_2 + R_3$$

$$\frac{s+1}{s(s^2+s-6)} = \frac{s+1}{s(s-2)(s+3)}$$

$$R_1 = \lim_{s\to 0}\left[s\frac{s+1}{s(s-2)(s+3)}e^{0t}\right] = \lim_{s\to 0}\left[\frac{s+1}{(s-2)(s+3)}\right] = -\frac{1}{6}$$

$$R_2 = \lim_{s\to 2}\left[(s-2)\frac{s+1}{s(s-2)(s+3)}e^{2t}\right] = \lim_{s\to 2}\left[\frac{s+1}{s(s+3)}e^{2t}\right] = \frac{3}{10}e^{2t}$$

$$R_3 = \lim_{s\to -3}\left[(s+3)\frac{s+1}{s(s-2)(s+3)}e^{-3t}\right] = \lim_{s\to -3}\left[\frac{s+1}{s(s-2)}e^{-3t}\right] = -\frac{2}{15}e^{-3t}$$

$$L^{-1}\left\{\frac{s+1}{s(s^2+s-6)}\right\} = R_1 + R_2 + R_3 = -\frac{1}{6} + \frac{3}{10}e^{2t} - \frac{2}{15}e^{-3t}$$

3)

$$L^{-1}\left\{\frac{2s^2+6s-4}{s(s+2)^2}\right\} = ?$$

$$L^{-1}\left\{\frac{2s^2+6s-4}{s(s+2)^2}\right\} = R_1 + R_2$$

$$R_1 = \lim_{s\to 0}\left[s\frac{2s^2+6s-4}{s(s+2)^2}e^{0t}\right] = -1$$

$$R_2 = \lim_{s\to -2}\frac{1}{(2-1)!}\frac{d}{ds}\left[(s+2)^2\frac{2s^2+6s-4}{s(s+2)^2}e^{-2t}\right] =$$

$$= \lim_{s\to -2}\left[\left(2+\frac{4}{s^2}\right)e^{st} + te^{st}\left(2s+6-\frac{4}{s}\right)\right]$$

$$= \left(2+\frac{4}{4}\right)e^{-2t} + te^{-2t}(-4+6+2)$$

$$= 3e^{-2t} + 4te^{-2t}$$

$$R_1 + R_2 = -1 + 3e^{-2t} + 4te^{-2t}$$

4)

$$L^{-1}\left\{\frac{1}{(s-3)^2(s-1)2}\right\}=?$$

$$L^{-1}\left\{\frac{1}{(s-3)^2(s-1)2}\right\}=R_1+R_2$$

$$R_1=\lim_{s\to 3}\frac{1}{(2-1)!}\frac{d}{ds}\left[(s-3)^2\frac{1}{(s-3)^2(s-1)^2}e^{st}\right]=$$

$$R_1=\lim_{s\to 3}\frac{d}{ds}\left[\frac{1}{(s-1)^2}e^{st}\right]=\lim_{s\to 3}\frac{-2}{(s-1)^3}e^{st}+\frac{1}{(s-1)^2}te^{st}$$

$$R_1=\lim_{s\to 3}\frac{-2}{(3-1)^3}e^{3t}+\frac{1}{(3-1)^2}te^{3t}=-\frac{1}{4}e^{3t}+\frac{1}{4}te^{3t}$$

$$R_2=\lim_{s\to 1}\frac{1}{(2-1)!}\frac{d}{ds}\left[(s-1)^2\frac{1}{(s-3)^2(s-1)^2}e^{st}\right]=$$

$$R_2=\lim_{s\to 1}\frac{d}{ds}\left[\frac{1}{(s-3)^2}e^{st}\right]=\lim_{s\to 1}\frac{-2}{(s-3)^3}e^{st}+\frac{1}{(s-3)^2}te^{st}$$

$$R_2=\frac{1}{4}e^{t}+\frac{1}{4}te^{t}\Rightarrow$$

$$L^{-1}\left\{\frac{1}{(s-3)^2(s-1)2}\right\}=R_1+R_2=-\frac{1}{4}e^{3t}+\frac{1}{4}te^{3t}+\frac{1}{4}e^{t}+\frac{1}{4}te^{t}$$

Heaviside Formula

$$\frac{P(s)}{Q(s)}=\frac{A_1}{(s-\alpha_1)}+\frac{A_2}{(s-\alpha_2)}+\frac{A_3}{(s-\alpha_3)}+\cdots+\frac{A_n}{(s-\alpha_n)}$$

$$L^{-1}\left\{\frac{P(s)}{Q(s)}\right\}=\frac{P(\alpha)}{Q'(\alpha)}e^{\alpha t}+\frac{P(\alpha_2)}{Q'(\alpha_2)}e^{\alpha_2 t}+\frac{P(\alpha_3)}{Q'(\alpha_3)}e^{\alpha_3 t}+\cdots+\frac{P(\alpha_n)}{Q'(\alpha_n)}e^{\alpha_n t}$$

1)

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\}=?$$

$$P(s)=s+3 \quad , \quad Q(s)=(s+1)(s-3)$$

$$Q'(s)=(s-3)+(s+1)$$

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\}=\frac{P(-1)}{Q'(-1)}e^{-t}+\frac{P(3)}{Q'(3)}e^{3t}$$

$$L^{-1}\left\{\frac{s+3}{(s+1)(s-3)}\right\}=\frac{2}{-4}e^{-t}+\frac{6}{4}e^{3t}=-\frac{1}{2}e^{-t}+\frac{3}{2}e^{3t}$$

2)

$$L^{-1}\left\{\frac{19s+37}{(s-1)(s+1)(s+3)}\right\}=?$$

$$P(s)=19s+37,\ P(1)=56,\ P(-1)=18,\ P(-3)=-20$$

$$Q(s)=(s-1)(s+1)(s+3)$$

$$Q'(s)=(s+1)(s+3)+(s-1)\big[(s+3)+(s+1)\big]$$

$$Q'(s)=(s+1)(s+3)+(s-1)(2s+4)$$

$$Q'(1)=8,\ \mathrm{Q}'(-1)=-4,\ \mathrm{Q}'(-3)=-8$$

$$L^{-1}\left\{\frac{19s+37}{(s-1)(s+1)(s+3)}\right\}=\frac{P(1)}{Q'(1)}e^{t}+\frac{P(-1)}{Q'(-1)}e^{-t}+\frac{P(-3)}{Q'(-3)}e^{-3t}$$

$$L^{-1}\left\{\frac{19s+37}{(s-1)(s+1)(s+3)}\right\}=\frac{56}{8}e^{t}+\frac{18}{-4}e^{-t}+\frac{-20}{-8}e^{-3t}$$

3)

$$L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}=?$$

$$L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\}=L^{-1}\left\{\frac{3s+1}{(s-1)(s+i)(s-i)}\right\}=?$$

$$P(s) = 3s+1,\ P(1) = 4,\ P(i) = 3i+1,\ P(-i) = -3i+1$$

and

$$Q(s) = (s-1)(s+i)(s-i) \Rightarrow Q'(s) = (s+i)(s-i) + 2s(s-1)$$

$$Q'(1) = 2,\ Q'(i) = 2i(i-1) = -2-2i,\ Q'(-i) = -2i(-i-1) = -2+2i$$

So,

$$L^{-1}\left\{\frac{3s+1}{(s-1)(s^2+1)}\right\} = \frac{P(1)}{Q'(1)}e^t + \frac{P(i)}{Q'(i)}e^{it} + \frac{P(-i)}{Q'(-i)}e^{-it}$$

$$= \frac{4}{2}e^t + \frac{3i+1}{-2-2i}e^{it} + \frac{-3i+1}{-2+2i}e^{-it}$$

Convolution

Another important general property of the Laplace transform has to do with products of transforms. It often happens that we are given two transforms *F(s)* and *G(s)* and whose inverses *f(t)* and *g(t)* we know, and we would like to calculate the inverse of the product *H(s)=F(s)G(s)* from those known inverses *f(t)* and *g(t)*. This inverse *h(t)* is written *(f*g)(t)*, which is a standard notation., and is called the convolution of *f* and *g*. How can we find h from *f* and *g*? This is stated in the convolution theorem. Since the situation and task just described arise quite often in applications, this theorem is considerable practical importance.

Convolution Theorem

Let *f(t)* and *g(t)* satisfy the hypothesis of existence theorem. Then the product of their transforms *F(s)=L{f(t)}* and *G(s)=L{g(t)}* is the transform *H(s)=L{h(t)}* of the convolution if *f(t)* and *g(t)*, written *L{f*g(t)}* and defined by

$$h(t) = f * g(t) = \int_0^t f(\tau)g(t-\tau)d\tau$$

Example 1:

Using convolution let's find the inverse *h(t)* of

$$H(s)=\frac{1}{(s^2+1)^2}=\frac{1}{(s^2+1)}\frac{1}{(s^2+1)}$$

We know that $L^{-1}\{\frac{1}{(s^2+1)}\}=\sin t$. By the convolution theorem we get

$$h(t)=L^{-1}\{H(s)\}=\sin t*\sin t=\int_0^t \sin\tau\sin(t-\tau)d\tau$$
$$=-\frac{1}{2}\int_0^t \cos t d\tau+\frac{1}{2}\int_0^t \cos(2\tau-t)d\tau$$
$$=-\frac{1}{2}\cos t+\frac{1}{2}\sin t$$

Example 2:

$1/s^2$ has the inverse t and $1/s$ had the inverse 1, and the convolution theorem confirms that

$$\frac{1}{s^3}=\left(\frac{1}{s^2}\right)\left(\frac{1}{s}\right)$$

has the inverse

$$t*1=\int_0^t \tau.1d\tau=\frac{t^2}{2}$$

Example 3:

Let

$$H(s)=\frac{1}{s^2(s-a)}$$

Find $h(t)$.

We know that

$$L\left\{\frac{1}{s^2}\right\}=t\quad L^{-1}\left\{\frac{1}{s-a}\right\}=e^{at}$$

Using the convolution theorem and integrating by parts, we get the answer.

$$h(t)=t*e^{at}=\int_o^t \tau e^{a(t-\tau)}d\tau=e^{at}\int_o^t \tau e^{-a\tau}d\tau$$
$$=\frac{1}{a^2}\left(e^{at}-at-1\right)$$

Differential Equations, Initial Value Problems

We consider an initial value problem

$$y''+ay'+by=r(t)\,, \qquad y(0)=M\,, \qquad y'(0)=N$$

with constant a and b. Here *r(t)* is the input (driving force) applied to the system and *y(t)* is the output (response of the system). In Laplace method there are three steps.

1st Step. Making a Laplace transform in both sides of the equation. This gives:

$$\left[s^2Y-sy(0)-y'(0)\right]+a\left[sY-y(0)\right]+bY=R(s)$$

and is called the **subsidiary equation**.

Collecting Y terms we have

$$Y(s^2+as+b)-sy(0)-y'(0)-ay(0)=R(s)$$
$$Y(s^2+as+b)=R(s)+sy(0)+y'(0)+ay(0)$$

2nd Step. Division by and use of the so-called transfer function

$$Q_s=\frac{1}{s^2+as+b}$$

gives as the solution of the subsidiary equation

$$Y(s)=\left[(s+y)y(0)+y'(0)\right]Q(s)+R(s)Q(s)$$

if $y(0)=y'(0)=0$, this simply $Y=RQ$; thus Q is the quotient

$$Q=\frac{Y}{S}=\frac{L(output)}{L(input)}$$

and this explains the name of Q. Note that Q depends only on a and b, but neither on r(t) nor on the initial conditions.

3rd Step. We reduce $Y(s)=[(s+y)y(0)+y'(0)]Q(s)+R(s)Q(s)$ equation (usually by partial fractions, as in integral calculus) to a sum of terms whose inverses can be found from the table, so that the solution $y(t)=L^{-1}\{Y\}$ of

the differential equation $y''+ay'+by=r(t)$ is obtained.

Example:

Let's solve the differential equation

$$y''-y=t\ ,\quad y(0)=1, y'(0)=1$$

1st Step: We use Laplace transform get the subsidiary equation

$$s^2Y-sy(0)-y'(0)-Y=\frac{1}{s^2}$$

and collecting *Y* terms

$$(s^2-1)Y=\frac{1}{s^2}+s+1$$

2nd Step: The transfer function $Q=\dfrac{1}{s^2-1}$. So,

$$Y=(s+1)Q+\frac{1}{s^2}Q=\frac{s+1}{s^2-1}+\frac{1}{s^2(s^2-1)}$$

$$=\frac{1}{s-1}+\left(\frac{1}{s^2-1}-\frac{1}{s^2}\right)$$

3rd Step: We take inverse Laplace transform of both sides of the equation.

$$L^{-1}(Y)=y(t)=e^t+\sinh t-t$$

The diagram summarizes the approach.

t-space		**s-space**
Given Problem $y''-y=t,\quad y(0)=1, y'(0)=1$	$\longrightarrow$	Subsidiary Equation $(s^2-1)Y=\frac{1}{s^2}+s+1$
		$\downarrow$
Solution of the given problem $y(t)=e^t+\sinh t-t$	$\longleftarrow$	Solution of the subsidiary Equation $Y=\frac{1}{s-1}+\frac{1}{s^2-1}-\frac{1}{s^2}$

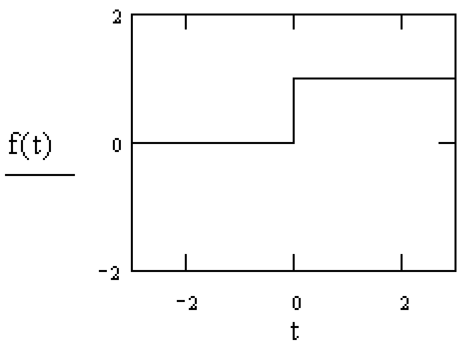
Unit Step Function

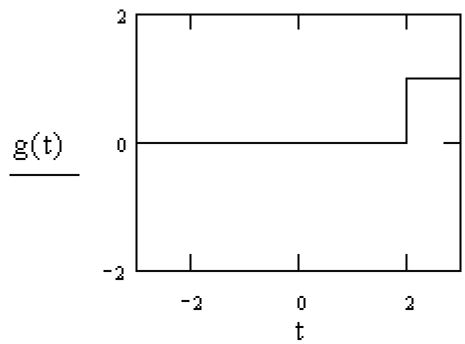
By definition $u(t-a)$ is 0 for $t<0$, has a jump of size 1 at $t=a$ (where we can leave it undefined) and is as for $t>a$

$$u(t-a)=\begin{cases}0, & if\ t<a\\ 1, & if\ t>a\end{cases}$$

If we use MathCad,

$\mathbf{f(t)} := \Phi(\mathbf{t})$ $\qquad$ $\mathbf{g(t)} := \Phi(\mathbf{t} - 2)$





Notice that the symbol for unit step function in MathCad is Φ.

Unit step function is also called Heaviside function. We can use unit step function to write *f(t)*

$$f(t)=\begin{cases}0, \text{ if } & t<a\\ f(t-a), \text{ if } & t>a\end{cases}$$

in the form $f(t-a)u(t-a)$, that is

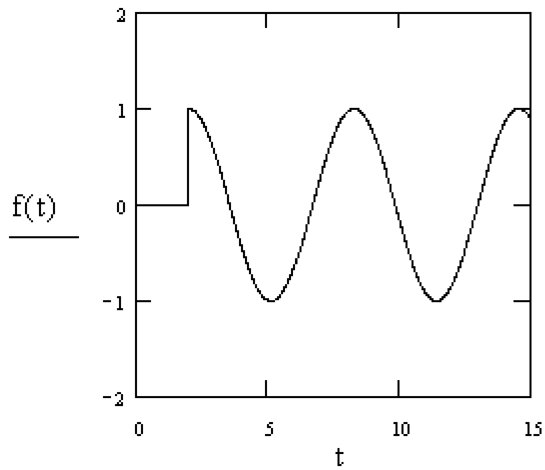
$$f(t-a)u(t-a)=\begin{cases}0, & \text{if } t<a\\ f(t-a), & \text{if } t>a\end{cases}$$

Let's take an example. *f(t)=cost for t>0*. And the *curve*

$$f(t-2)u(t-2)=\cos(t-2)u(t-2)$$

obtained by shifting it 2 units to the right. For $t \le a = 2$, this function is zero because *u(t-2)* has this property.

$$\mathbf{f(t) := cos(t - 2)\ \Phi(t - 2)}$$



Laplace of Unit-Step Function

If $L\{f(t)\} = F(s)$, then

$$L\{f(t-a)u(t-a)\} = e^{-as}F(s) \quad (1)$$

Taking the inverse transform on both sides of this equation and interchanging side we obtain the companion formula

$$L^{-1}\{e^{-as}F(s)\} = f(t-a)u(t-a)$$

Proof of Second Shifting Theorem:

From the definition we have

$$e^{-as}F(s) = e^{-as}\int_0^\infty e^{-s\tau} f(\tau)d\tau = \int_0^\infty e^{-s(\tau+a)} f(\tau)d$$

Substituting $\tau + a = t$ int the integral, we obtain

$$e^{-as}F(s) = \int_a^\infty e^{-st} f(t-a)dt$$

We can write this as an integral from zero to infinity if we make sure that the integrand is zero for all *t* from *0* to *a*. We may easily accomplish this by multiplying the present integrand by the step function *u(t-a)*, thereby completing the proof.

$$e^{-as}F(s) = \int_0^{\infty} e^{-st} f(t-a)u(t-a)dt = L\{f(t-a)u(t-a)\}$$

In fact, we should see the transform of unit-step function as well. Directly it is:

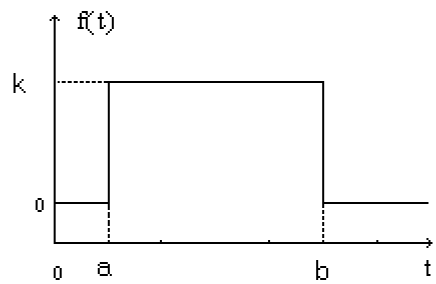
$$L\{u(t-a)\} = \frac{e^{-as}}{s}$$

This formula follows directly from the definition.

$$L\{u(t-a)\} = \int_o^{\infty} e^{-st}u(t-a)dt = \int_o^{a} e^{-st}0dt + \int_a^{\infty} e^{-st}1dt = -\frac{1}{s}e^{-st}\Big|_a^{\infty} = \frac{e^{-st}}{s}$$

Example 1:

Find the Laplace transform of the following function.



Solution:

By analyzing the graphic we can say

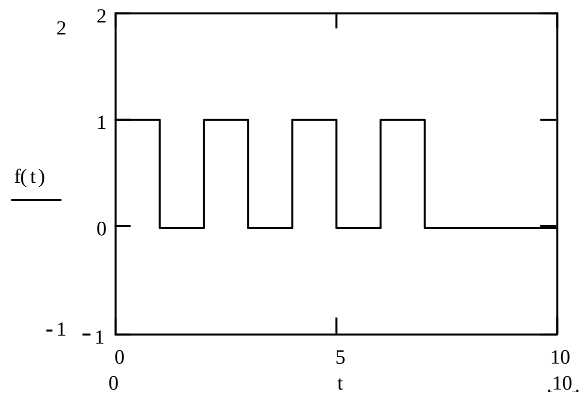
$$f(t) = \begin{cases} 0, & t < a \\ k, & \text{a} < t < b \\ 0, & t > b \end{cases}$$

$$\Rightarrow \quad f(t) = ku(t-a) - ku(t-b)$$

$$L\{f(t)\} = \frac{ke^{-as}}{s} - \frac{ke^{-bs}}{s}$$

Example 2:

Find the Laplace transform of the following function.



Solution:

From the graphic we can say

$$f(t) = u(t) - u(t-1) + u(t-2) - u(t-3) + u(t-4) - u(t-5) + \cdots + u(t-n) - u(t-(n+1))$$

$$L\{f(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s} - \frac{e^{-3s}}{s} + \frac{e^{-4s}}{s} - \frac{e^{-5s}}{s} + \cdots + \frac{e^{-ns}}{s} - \frac{e^{-(n+1)s}}{s}$$

Dirac's Delta Function

Phenomena of an impulsive nature, such as the action of very large forces (or voltages) over very short intervals of time, are of great practical interest, since they arise in various practical applications. This situation occurs, for instance, when a tennis ball is hit, a system is given a blow by a hammer, an airplane makes a hard landing, a ship is hit by a high single wave, and so on. Our present goal is to solve problems involving short impulses by Laplace transforms.

In mechanics, the impulse of a force *f(t)* over a time interval, say, $a \le t \le a+k$ is defined to be the integral of *f(t)* from a to $a+k$.

The analog for an electric circuit is the integral of the electromotive force applied to the circuit, integrated from a to a+k. Of particular practical interest is the case of a very short k (and its limit $k \to 0$), that is, the impulse of a force acting only for an instant. To handle the case, we consider the function

$$f_k(t) = \begin{cases} \frac{1}{k} & if \quad a \le t \le a+k \\ 0 & otherwise \end{cases}$$

Its impulse I_k is 1, since the integral evidently gives the area of the rectangle.

$$I_k = \int_0^\infty f_k(t)dt = \int_a^{a+k} \frac{1}{k} dt = 1$$

We can represent this function in terms of two unit step functions.

$$f_k(t) = \frac{1}{k}\left[u(t-a) - u(t-(a+k))\right]$$

Recalling the Laplace transform of unit-step function, we obtain the Laplace transform

$$L\{f_k(t)\} = \frac{1}{ks}(e^{-as} - e^{-a(+k)s}) = e^{-as}\frac{1-e^{-(a+k)s}}{ks}$$

The limit of this function as k goes to 0 is denoted by $\delta(t-a)$

and is called the Dirac delta function (sometimes the unit impulse function).

Now, let' turn to the following equation again.

$$L\{f_k(t)\} = \frac{1}{ks}(e^{-as} - e^{-a(+k)s}) = e^{-as}\frac{1-e^{-(a+k)s}}{ks}$$

The quotient in this equation has the limit 1 as k goes to 0, as follows by l'Hospital's rule.(Differentiate the numerator and the denominator at the same time with respect to k)

Thus,

$$L\{\delta(t-a)\} = e^{-as}$$

We note that Dirac delta function is not a function in the ordinary sense as used in calculus, but a so-called generalized function.

$$\delta(t-a) = \begin{cases} \infty & if \quad t = a \\ 0 & otherwise \end{cases} \qquad \text{and} \quad \int_0^\infty \delta(t-a)dt = 1$$

But an ordinary function which is everywhere 0 except at a single point must have the integral 0. Nevertheless, in impulse problems it is convenient to operate on Dirac delta function as if it were an ordinary function.

Example: Response of a damped vibrating system to a unit impulse**.**

Determine the response of a damped mass-spring system governed by

$$y''+3y'+2y=\delta(t-a)\,, \qquad y(0)=0,\ \ y'(0)=0$$

Thus the system is initially at rest and the time $t=a$ is suddenly given a sharp hammerblow.

Solution:

Taking Laplace transform we obtain

$$s^2Y+3sY+2Y=e^{-as}$$

Solving for Y, we have

$$Y(s)=F(s)e^{-as}\text{, where } F(s)=\frac{1}{(s+1)(s+2)}=\frac{1}{s+1}-\frac{1}{s+2}$$

Taking the inverse transform we obtain

$$f(t)=e^{-t}-e^{-2t}\,.$$

Hence by the second shifting theorem we have

$$y(t)=L^{-1}\{e^{-as}F(s)\}=f(t-a)u(t-a)=\begin{cases}0, & if \quad 0\le t\le a\\ e^{-(t-a)}-e^{-2(t-a)}, & if \quad t>a\end{cases}$$

SECTION 2

- Pre-Information

- Laplace Transforms in Electric Circuit Analysis

The Capacitor

The current through capacitor is proportional to the rate at which the voltage across the capacitor varies with time, or, mathematically,

$$i = C\frac{dv}{dt}$$

This gives the capacitor current as a function of the capacitor voltage. Expressing the voltage as a function of current is also useful. To do so,

$$idt = Cdv$$

$$\int_{v(t_0)}^{v(t)} dv = \frac{1}{C}\int_{t_0}^{t} idt$$

$$v(t) = \frac{1}{C}\int_{t_0}^{t} idt + v(t_0)$$

We can derive the power and energy relationships for the capacitor. From the definition of power

$$p = vi = Cv\frac{dv}{dt}$$

or

$$p = i\left[\frac{1}{C}\int_{t_0}^{t} idt + v(t_0)\right]$$

Combining the definition of energy with the first power equation above

$$p = \frac{dw}{dt} = Cv\frac{dv}{dt}$$
$$dw = Cvdv$$
$$\int_0^w dw = C\int_o^v vdv$$
$$w = \frac{1}{2}Cv^2$$

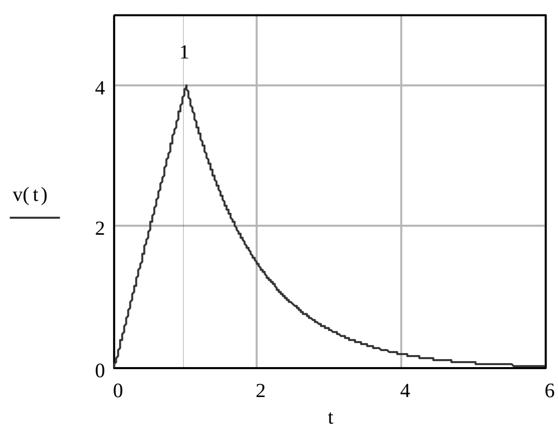
Example: The voltage pulse described by the following equation is impressed across the terminals of a 0.5uF capacitor.

$$v(t) = 0, \qquad t \le 0;$$
$$v(t) = 4t \;V, \qquad 0 \le t \le 1;$$
$$v(t) = 4e^{-(t-1)} \;V, \qquad 1 \le t \le \infty.$$

Now, we will derive the expressions for the capacitor current, power, and energy and sketch the graphs.

First of all, if we sketch the voltage graph using MathCad we get the following result. Here we used unit step function to express all the intervals in one step.

$$v(t) := 4\cdot t\cdot(\Phi(t) - \Phi(t-1)) + 4\cdot e^{-(t-1)}\,\Phi(t-1)$$



a) Using the following formula we can derive the current equation from voltage equation.

$$i(t) = C\frac{dv}{dt}$$

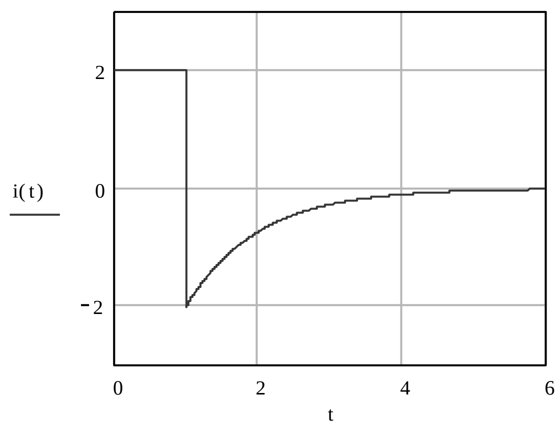
$$i(t) = 0 \qquad \text{for } t \le 0$$

$$i(t) = C\frac{dv}{dt} = (0.5\times10^{-6})\frac{d(4t)}{dt} = 0.5\mu F \times 4 = 2\mu A \qquad \text{for } 0 \le t \le 1$$

$$i(t) = C\frac{dv}{dt} = (0.5\times10^{-6})\frac{d(4e^{-(t-1)})}{dt} = -2e^{-(t-1)}\mu A \qquad \text{for } 1 \le t \le \infty$$

And the graph of this current is seen below.

$$i(t) := 2\cdot(\Phi(t) - \Phi(t-1)) - 2\cdot e^{-(t-1)}\,\Phi(t-1)$$



b) Now let’s find the power equation. If we remember the formula

$$p = vi = Cv\frac{dv}{dt},$$

we can easily derive the power.

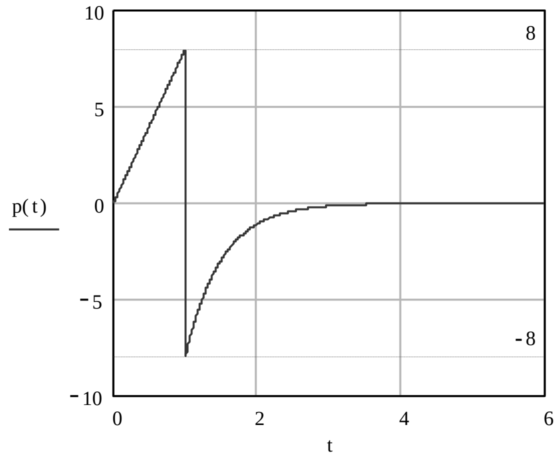
$$p = 0 \qquad \text{for } t \le 0$$

$$p = 4t \times i(t) = 4t \times 2 = 8t\mu W \qquad \text{for } 0 \le t \le 1$$

$$p = (4e^{-(t-1)})(-2e^{-(t-1)}) = -8e^{-2(t-1)}\mu W \qquad \text{for } 1 < t \le \infty$$

So, the power graph:

$$p(t) := 8 \cdot t \cdot (\Phi(t) - \Phi(t-1)) - 8 \cdot e^{-2 \cdot (t-1)} \Phi(t-1)$$



c) And the energy equation can be derived from the following equation.

$$w = \frac{1}{2} C v^2$$

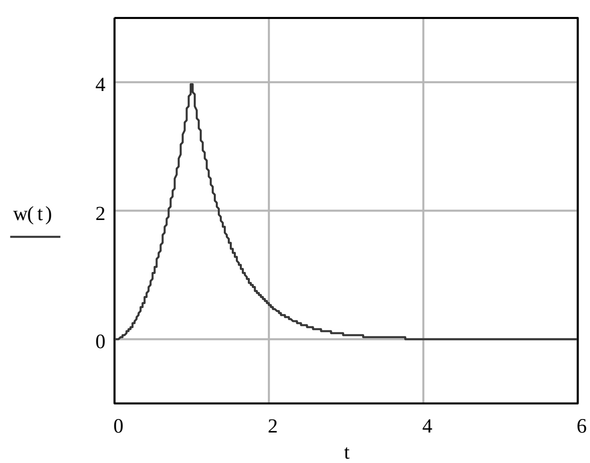
$$w = 0 \qquad for\ t \le 0$$

$$w = \frac{1}{2}(0.5)16t^2 = 4t^2 \mu J \qquad for\ 0 < t \le 1$$

$$w = \frac{1}{2}(0.5)16e^{-2(t-1)} = 4e^{-2(t-1)} \mu J \quad for\ 1 < t \le \infty$$

Finally, the energy graph is

$$w(t) := 4 \cdot t^2 \cdot (\Phi(t) - \Phi(t-1)) + 4 \cdot e^{-2 \cdot (t-1)} \Phi(t-1)$$



Energy is being stored in the capacitor whenever the power is positive. Hence energy is being stored in the interval 0 to 1 s.

Energy is being delivered by the capacitor whenever is the power is negative. Thus energy is being delivered for all t greater than 1 s.

Inductor

A current change through an inductor results in a voltage on the inductor and this voltage is given by

$$v = L\frac{di}{dt}$$

v is measured in volts, L in henrys, I in amperes, and t in seconds.

Current in an Inductor in Terms of Voltage

$$v = L\frac{di}{dt}$$

$$vdt = L\left(\frac{di}{dt}\right)dt$$

$$vdt = Ldi$$

$$L\int_{i(t_0)}^{i(t)} dx = \int_{t_0}^{t} v\,d\tau$$

$$i(t) = \frac{1}{L}\int_{t_0}^{t} v\,d\tau + i(t_0)$$

In many practical applications to is zero so the equation becomes

$$i(t) = \frac{1}{L}\int_0^t v\,d\tau + i(0)$$

Power in an Inductor

$$p = vi$$
$$p = Li\frac{di}{dt}$$

We can also express the current in terms of voltage

$$p = v\left[\frac{1}{L}\int_{t_0}^t v\,d\tau + i(t_0)\right]$$

But the first equation is most useful in expressing the energy stored in the inductor.

Energy in an Inductor

Power is the time rate of expending energy, so

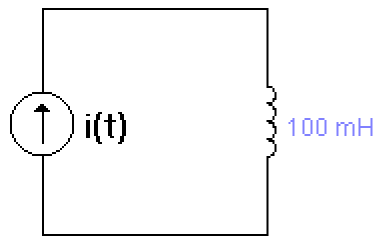
$$p = \frac{dw}{dt} = Li\frac{di}{dt}$$
$$dw = Li\,di$$
$$\int_0^w dx = L\int_0^i y\,dy$$
$$w = \frac{1}{2}Li^2$$

We use different symbols of integration to avoid confusion with the limits placed on the integrals. The energy is in joules when inductance is in henrys and current is in amperes.

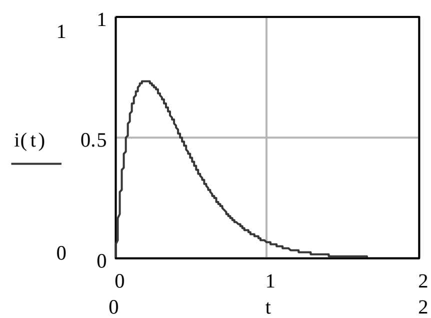
Example:

The independent current source in the circuit generates zero current for t<0 and a pulse $i(t) = 10te^{-5t}$ t>0.

a) Sketch the current waveform
b) At what instant of time is the current maximum?
c) Express the voltage across the terminals of the 100 mH inductor as a function of time.
d) Sketch the voltage waveform
e) At what instant of time does the voltage change polarity?
f) Find the power equation and sketch the power waveform.



a) The current waveform is shown below, which was obtained using MathCad.



b) If we take the derivative of *i(t)* and making what we get equal to zero, we find the maximum of the current.

$$\frac{di}{dt} = \frac{d(10te^{-5t)})}{dt} = 10(-5te^{-5t} + e^{-5t}) = 10e^{-5t}(1-5t)$$

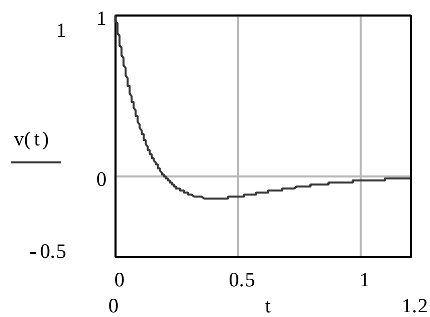
$$10e^{-5t}(1-5t) = 0$$

$$t = \frac{1}{5} = 0.2s$$

c)

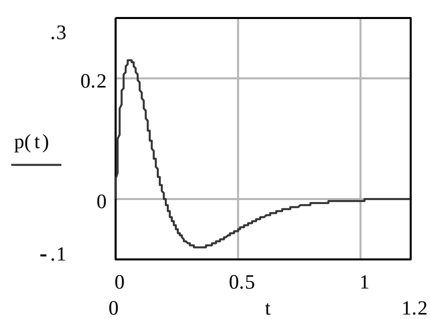
$$v = \mathrm{L}\frac{di}{dt} = (0.1)(10e^{-5t})(1-5t) = e^{-5t}(1-5t)\ \mathrm{V},\ t > 0 \quad v = 0, t < 0$$

d) Using MathCad, if we sketch $v(t) = e^{-5t}(1-5t)$ we get the following result.

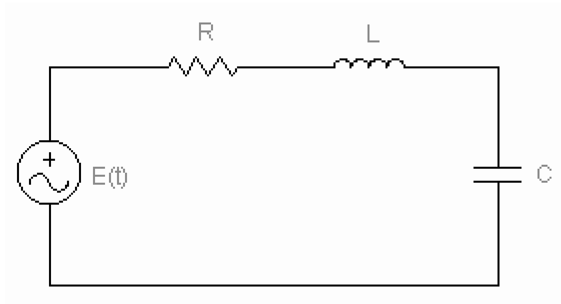


e) If we notice, we see that at t=0.2 s the voltage changes polarity. This is the instant when current is maximum.

f) $p(t) = i(t).v(t) = 10te^{-10t} - 50t^2e^{-10t}$ W



Series RLC Circuit



In the RLC circuit in the figure above, the voltage drops across the inductor, resistor, and capacitor are given by $L\frac{di}{dt}$, Ri, and $\frac{1}{C}\int_0^t i(\tau)d\tau$

Kirchhoff's voltage law states that the sum of voltage drops across the individual components equals the impressed voltage, *E(t).* So,

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int_0^t i(\tau)d\tau = E(t)$$

Setting $Q(t) = \int_o^t i(\tau)d\tau$ (the charge of the condenser), and $i = \frac{dQ}{dt}$ we can write

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t)$$

So, this is the basis for this kind of electrical problems.

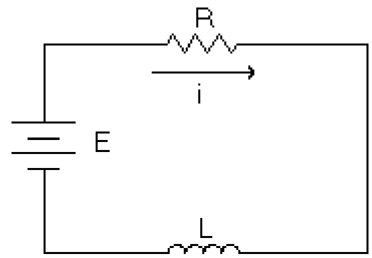
SECTION 3

Laplace Transforms in Electric Circuit Analysis

In this section, we will see how we face integro-differential equations in electric circuit analysis and how we will explain in examples how we solve those equations using Laplace transforms. In fact, solving those equations doesn't require a deep information about Laplace transforms. Having an idea about basic formulas would be enough. Interestingly, we will not need even those formulas in the last section to solve those problems.

Example 1: Finding general current equation for series RL circuit with a constant E

Let's take a look to the circuit seen below. We can easily write its voltage equation through Kirchhoff laws. And if we want to see its current equation, or in other words, if we want to see the behavior of current in this circuit in a mathematical way, the following mathematical operations will lead us to the answer.



$$E = \mathrm{L}\frac{di}{dt} + iR$$

Take the Laplace transform of both sides of this equation.

$$L\{E\} = L\left\{\mathrm{L}\frac{di}{dt}\right\} + L\{iR\}$$

$$\frac{E}{s} = \mathrm{L}\big(sI(s) + i(0)\big) + I(s)R$$

Assume $i(0) = 0$

$$\frac{E}{s} = s\mathrm{L}I(s) + I(s)R$$

$$\frac{E}{s} = I(s)(s\mathrm{L} + R)$$

$$I(s) = \frac{E}{s(s\mathrm{L} + R)}$$

$$I(s) = \frac{E}{\mathrm{L}s\left(s + \frac{R}{\mathrm{L}}\right)}$$

$$L^{-1}\{I\} = \frac{E}{\mathrm{L}}L^{-1}\left\{\frac{1}{s\left(s + \frac{R}{\mathrm{L}}\right)}\right\}$$

Use partial fraction

$$L^{-1}\{I\} = \frac{E}{\mathrm{L}}\left[L^{-1}\left\{\frac{A}{s}\right\} + L^{-1}\left\{\frac{B}{s + \frac{R}{\mathrm{L}}}\right\}\right] \quad (2)$$

A and B are found in the following way.

$$\frac{1}{s\left(s+\frac{R}{\mathrm{L}}\right)}=\frac{A}{s}+\frac{B}{s+\frac{R}{\mathrm{L}}}$$

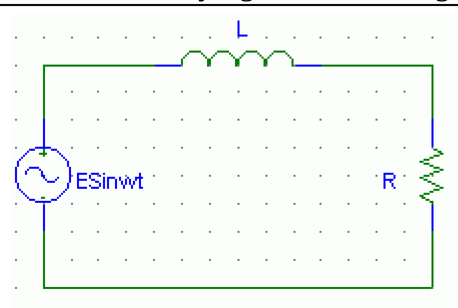
$$1=A\left(s+\frac{R}{\mathrm{L}}\right)+Bs \quad (3)$$

Make $s=0$ in (3) to annihilate B, then $A=\frac{\mathrm{L}}{R}$. Make $s=-\frac{R}{\mathrm{L}}$ in (3) to annihilate A, then $B=-\frac{\mathrm{L}}{R}$. Put the values of A and B into (2),

$$L^{-1}\{I\}=\frac{E}{\mathrm{L}}\left[\frac{\mathrm{L}}{\mathrm{R}}\left(L^{-1}\left\{\frac{1}{s}\right\}-L^{-1}\left\{\frac{1}{s+\frac{R}{\mathrm{L}}}\right\}\right)\right]$$

$$i(t)=\frac{E}{R}\left(1-e^{-\frac{R}{\mathrm{L}}t}\right)$$

Example 2: General Current Equation in series RL circuit with a varying $E_o$sinwt voltage



Suppose that the current i in an electrical circuit satisfies

$$\mathrm{L}\frac{di}{dt}+Ri=E_0\sin\omega t$$

where L,R,Eo and *w* constants. Find *i(t)* for *t>0* if *i(0)=0*. Taking the Laplace transform

$$\mathrm{L}sL\{i(t)\}+RL\{i(t)\}=\frac{E_0\omega}{s^2+\omega^2}$$

that is,

$$\boldsymbol{L\{i(t)\}=\frac{E_0\omega}{(\mathrm{L}s+R)(s^2+\omega^2)}}$$

Considering partial fractions

$$L\{i(t)\}=\frac{E_0\omega/\mathrm{L}}{(s+\frac{R}{\mathrm{L}})(s^2+\omega^2)}=\frac{A}{s+\frac{R}{\mathrm{L}}}+\frac{Bs+C}{s^2+\omega^2},$$

and we find that

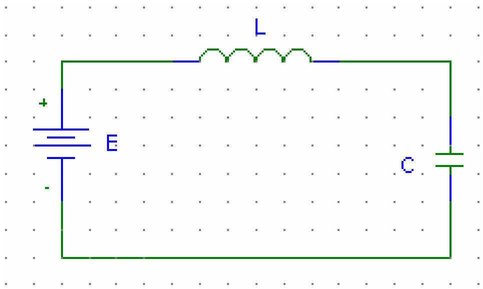
$$A=\frac{E_0\mathrm{L}\omega}{\mathrm{L}^2\omega^2+R^2},\qquad B=\frac{-E_0\mathrm{L}\omega}{\mathrm{L}^2\omega^2+R^2},\qquad \mathrm{C}=\frac{E_0\mathrm{R}\omega}{\mathrm{L}^2\omega^2+R^2},$$

and so

$$i(t)=\frac{E_0\mathrm{L}\omega}{\mathrm{L}^2\omega^2+R^2}e^{-\frac{R}{L}t}+\frac{E_0R}{\mathrm{L}^2\omega^2+R^2}\sin\omega t-\frac{E_0\mathrm{L}\omega}{\mathrm{L}^2\omega^2+R^2}\cos\omega t.$$

Example 3: Finding the General Current Equation in series LC circuit

Suppose that the current i in the electrical circuit beside satisfies



$$\mathrm{L}\frac{di}{dt}+\frac{1}{C}\int_0^t i(\tau)d\tau=E,$$

where L, C and E are positive constants, *i(0)=0*. Then

$$\mathrm{L}sL\{i(t)\}+\frac{L\{i(t)\}}{sC}=\frac{E}{s}$$

$$L\{i(t)\}\left(\mathrm{L}s+\frac{1}{sC}\right)=L\{i(t)\}\left(\frac{s^2\mathrm{L}C+1}{sC}\right)=\frac{E}{s}$$

implying

$$L\{i(t)\}=\frac{CE}{\mathrm{L}Cs^2+1}=\frac{E}{\mathrm{L}(s^2+1/\mathrm{L}C)}$$

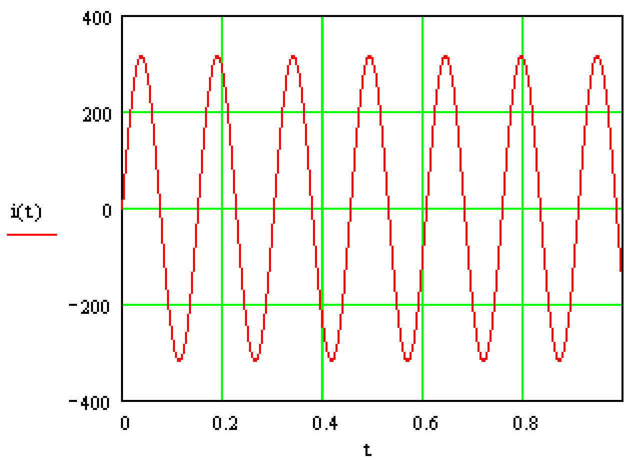
And inverse Laplace of both sides

$$i(t)=E\sqrt{\frac{\mathrm{L}}{C}}\sin\frac{1}{\sqrt{\mathrm{L}C}}t$$

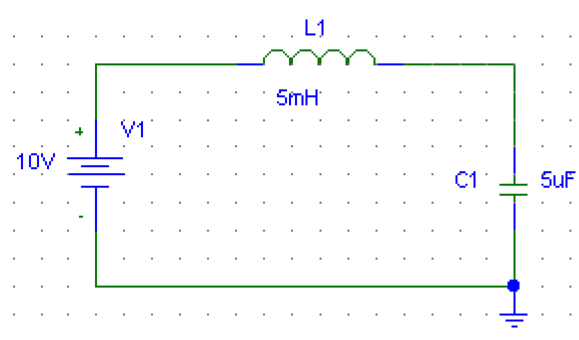
So, setting some values we can see the current behavior graphically. Let's see it theoretically in MathCad first and then compare the result with the one we obtained in PSpice.

$$E := 10 \qquad L := 5 \qquad C := 5$$

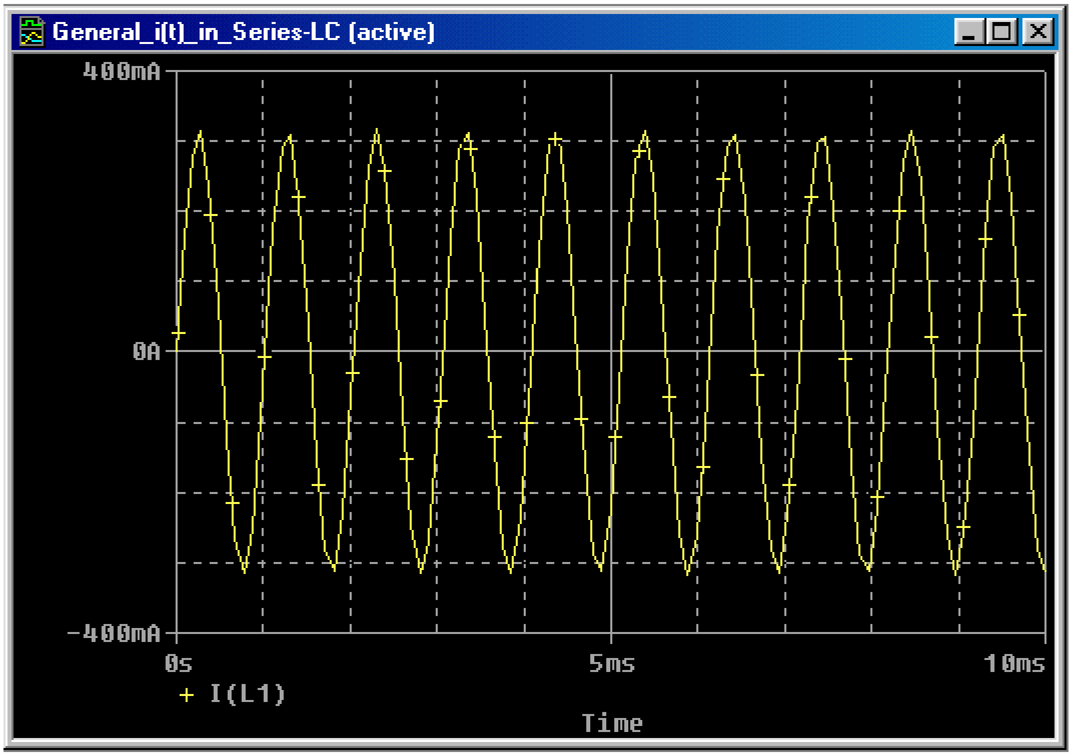
$$i(t) := 10 \cdot \sqrt{\frac{5 \cdot 10^{-3}}{5 \cdot 10^{-6}}} \cdot \sin\left(\frac{t}{\sqrt{5 \cdot 10^{-3} \cdot 5 \cdot 10^{-6}}}\right)$$



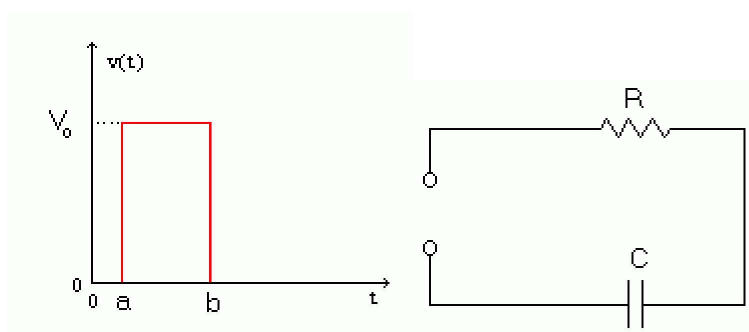
The circuit in PSpice is the following one.



And the current through inductor is seen below.



Example 4: Response of an RC-circuit to a single square wave



The equation of the circuit is

$$Ri(t) + \frac{q(t)}{C} = Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = v(t)$$

where *v(t)* can be represented in terms of two unit step functions.

$$v(t) = V_0\left[u(t-a) - u(t-b)\right]$$

Then we have

$$Ri(t) + \frac{1}{C}\int_0^t i(\tau)d\tau = V_0\left[u(t-a) - u(t-b)\right]$$

If we take Laplace transform of both sides of the equation

$$RI(s) + \frac{I(s)}{sC} = \frac{V_0}{s}(e^{-as} - e^{-bs})$$

$$I(s)\left(R + \frac{1}{sC}\right) = \frac{V_0}{s}(e^{-as} - e^{-bs})$$

$$I(s) = \frac{sC}{(sRC+1)}\frac{V_0}{s}(e^{-as} - e^{-bs})$$

$$I(s) = \frac{V_0 / R}{s + \frac{1}{RC}}(e^{-as} - e^{-bs})$$

Now, using inverse Laplace transform we can find *i(t)*.

$$i(t) = L^{-1}\left(\frac{V_0 / R}{s + \frac{1}{RC}}e^{-as}\right) - L^{-1}\left(\frac{V_0 / R}{s + \frac{1}{RC}}e^{-bs}\right)$$

If we remember that

$$L^{-1}\left(\frac{A}{s+B}e^{-as}\right) = Ae^{-B(t-a)}u(t-a)$$

So,

$$i(t) = \frac{V_0}{R}e^{-(t-a)/RC}u(t-a) - \frac{V_0}{R}e^{-(t-b)/RC}u(t-b)$$

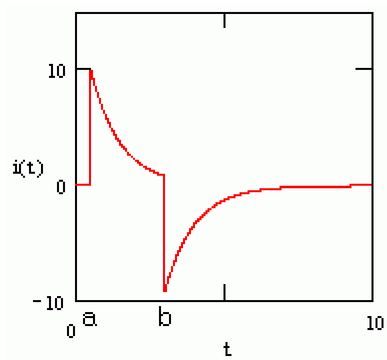
that is,

$$i(t) = \begin{cases} 0 & ,\ t < a \\ Me^{-t/RC} & ,\ a < t < b \\ (M-N)e^{-t/RC} & ,\ t > b \end{cases}$$

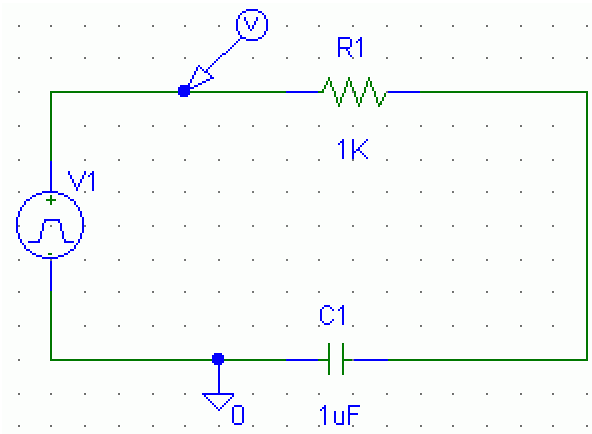
where

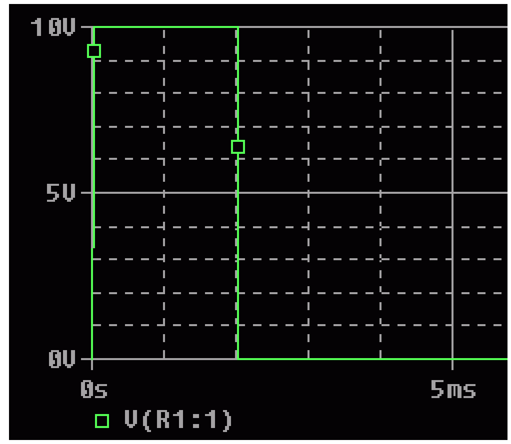
$$M = \frac{V_0}{R} e^{a/RC}$$

$$N = \frac{V_0}{R} e^{b/RC}$$

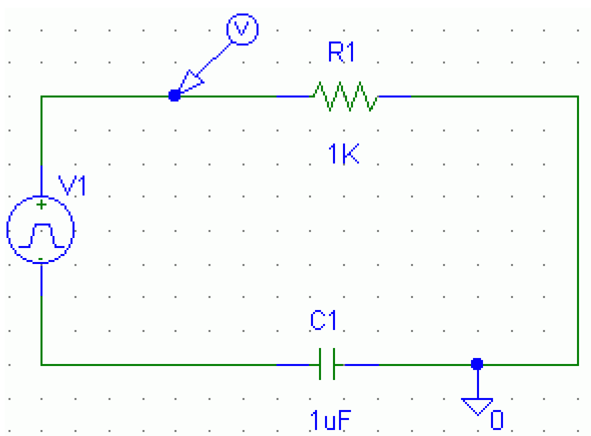


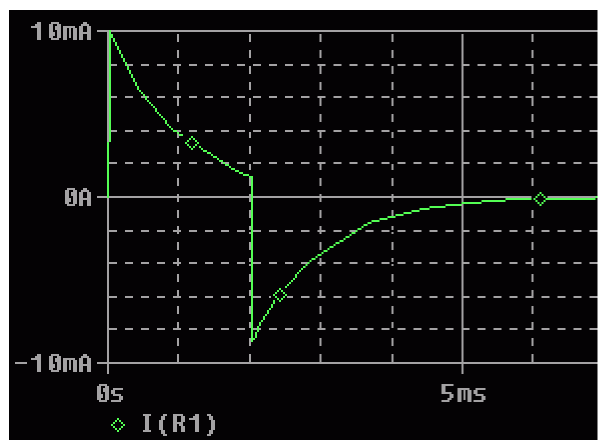
Verifying the mathematical result using PSpice.





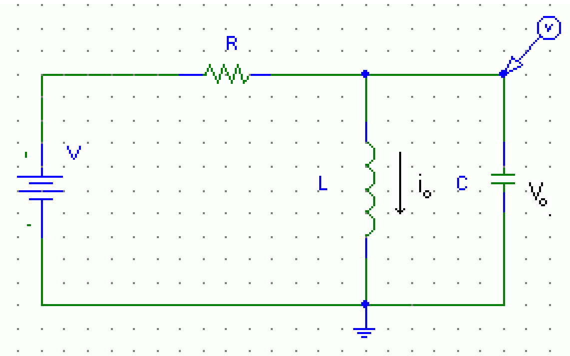
This is our circuit in PSpice. At first, we measure the input signal by putting the ground next to impulse generator. The input signal is seen on the right side.





So now, we are ready to measure the output signal. We change the place of the ground and get the response of the circuit to this input signal.

Example 5: Find voltage and current equations in R series to parallel LC circuit.



All the initial currents in this circuit is zero at *t=0*.

-Derive the integro differential equation that governs the behavior of the circuit.

-Show that $V_o(s) = \dfrac{V/RC}{s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}}$

-Show that $I_o(s) = \dfrac{V/RLC}{s\left(s^2 + \dfrac{1}{RC}s + \dfrac{1}{LC}\right)}$

-Find Vo(t) and io(t) for t>0 for the following values.

V=35 V, R=5Kohm, L=200mH, C=0.1uF

and plot the graph for those values in MathCad.

-Plot the graph of the circuit in PSpice and see if they are in accordance with those in MathCad.

-Derive the integro differential equation that governs the behavior of the circuit.

If we assume that the total current which flows through R is 'i'

$$V = Ri + v_o = R(i_o + i_C) + v_o$$

$$V = R(\frac{1}{L}\int_0^t v_o dt + \frac{Cdv_o}{dt}) + v_o$$

$$V = \frac{R}{L}\int_0^t v_o dt + \frac{RCdv_o}{dt} + v_o$$

-Find the Laplace transform of voltage equation.

$$\frac{V}{s} = \frac{RV_o(s)}{s\mathrm{L}} + RCsV_o(s) + V_o(s)$$

$$\frac{V}{s} = V_o(s)\left(\frac{R}{s\mathrm{L}} + RCs + 1\right) = V_o(s)\left(\frac{R + RLCs^2 + sL}{sL}\right)$$

$$V_o(s) = \frac{LV}{RLCs^2 + sL + R} = \frac{LV}{RLC\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}$$

$$V_o(s) = \frac{V / RC}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

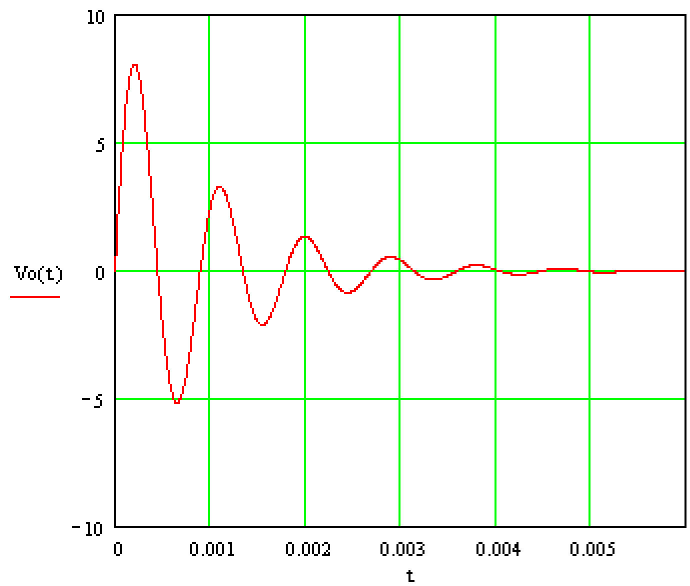
Find output voltage equation for the following values and plot the graph for those values in MathCad..

First we write *v(s)* for the specified values. Then following Symbolics->Transform->Inverse Laplace combination (make sure that the cursor is next to 's' during this operation) we get the result in t domain. Then using this result in our *Vo(t)* function, we plot the graph.

$$v(s) := \frac{7 \cdot 10^4}{s^2 + 2 \cdot 10^3 \cdot s + 5 \cdot 10^7}$$

$$10 \cdot \exp(-1000 \cdot t) \cdot \sin(7000 \cdot t)$$

$$Vo(t) := 10 \cdot \exp(-1000 \cdot t) \cdot \sin(7000 \cdot t)$$



Plot the voltage graph of the circuit in PSpice and see if it is in accordance with those in MathCad

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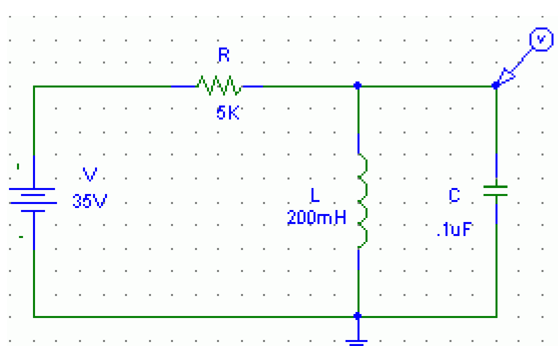
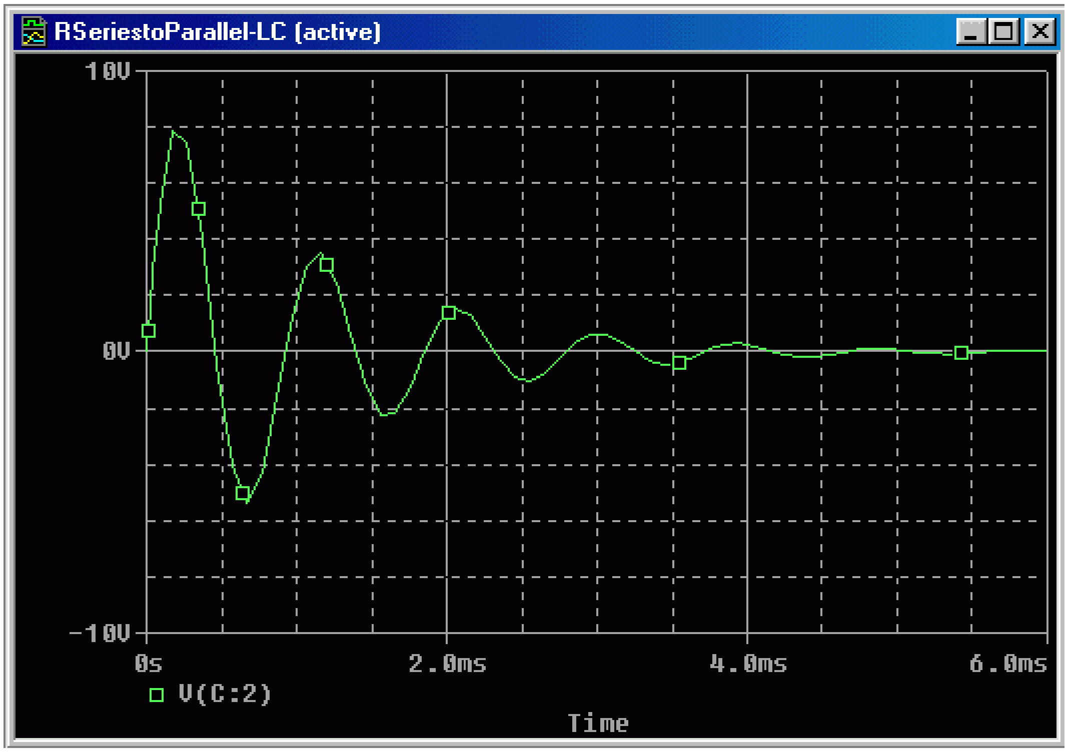
This is our circuit in PSpice with the specified values. So, for the values given, the output voltage graph we obtained inPspice is seen below. And it is total in accordance with what we obtained in MathCad in a pure mathematical approach.

Note: Notice that the value of capacitor is 0.1uF. In Pspice .1 and 0.1 are the same.



$v_{o(t)}$ of the circuit in PSpice:



General Solution for Vo(s)

Now, we will inverse Laplace this equation. We have to use partial fractions. But we have to separate the denominator first. If we find the roots of denominator;

$$\Delta = \left(\frac{1}{RC}\right)^2 - 4\left(\frac{1}{LC}\right)$$

$$s_1 = \frac{-\frac{1}{RC} - \sqrt{\Delta}}{2} = -\frac{1}{2RC} - \frac{\sqrt{\Delta}}{2}, \quad s_2 = \frac{-\frac{1}{RC} + \sqrt{\Delta}}{2} = -\frac{1}{2RC} + \frac{\sqrt{\Delta}}{2}$$

so,

$$V_o(s) = \frac{V}{RC}\left(\frac{1}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}\right) = \frac{V}{RC}\left(\frac{A}{s + \frac{1}{2RC} + \frac{\sqrt{\Delta}}{2}} + \frac{B}{s + \frac{1}{2RC} - \frac{\sqrt{\Delta}}{2}}\right)$$

here,

$$A = -\frac{1}{\sqrt{\Delta}}, \quad B = \frac{1}{\sqrt{\Delta}} \quad \text{so,}$$

$$V_o(s) = \frac{V}{RC\sqrt{\Delta}}\left(\frac{-1}{s + \frac{1}{2RC} + \frac{\sqrt{\Delta}}{2}} + \frac{1}{s + \frac{1}{2RC} - \frac{\sqrt{\Delta}}{2}}\right)$$

We can inverse Laplace the equation now.

$$v_o(t) = \frac{V}{RC\sqrt{\Delta}}\left(-e^{-\left(\frac{1}{2RC}+\frac{\sqrt{\Delta}}{2}\right)t} + e^{-\left(\frac{1}{2RC}+\frac{-\sqrt{\Delta}}{2}\right)t}\right) = \frac{V}{RC\sqrt{\Delta}}\left(-e^{\frac{-t}{2RC}}e^{\frac{-\sqrt{\Delta}}{2}t} + e^{\frac{-t}{2RC}}e^{\frac{\sqrt{\Delta}}{2}t}\right)$$

$$v_o(t) = \frac{V}{RC\sqrt{\Delta}}e^{\frac{-t}{2RC}}\left(e^{\frac{\sqrt{\Delta}}{2}t} - e^{\frac{-\sqrt{\Delta}}{2}t}\right) = \frac{V}{RC\sqrt{\Delta}}e^{\frac{-t}{2RC}}\sin\frac{\sqrt{\Delta}}{2}t$$

Find the current *Io(s)*

The total voltage is the sum of the voltage on R and on the parallel LC branch. We can express total current in terms of two currents. The current which flows through L and the current which flows through C

$$V = L\frac{di_o}{dt} + Ri = L\frac{di_o}{dt} + R\left(i_o + \frac{Cdv_o}{dt}\right)$$

Since they are parallel to each other, the voltages on C and L are equal

$$v_o = L\frac{di_o}{dt} \Rightarrow$$

$$V = L\frac{di_o}{dt} + R\left(i_o + \frac{Cd\left(\frac{Ldi_o}{dt}\right)}{dt} \right) = L\frac{di_o}{dt} + R\left(i_o + \frac{LCd^2 i}{dt^2} \right)$$

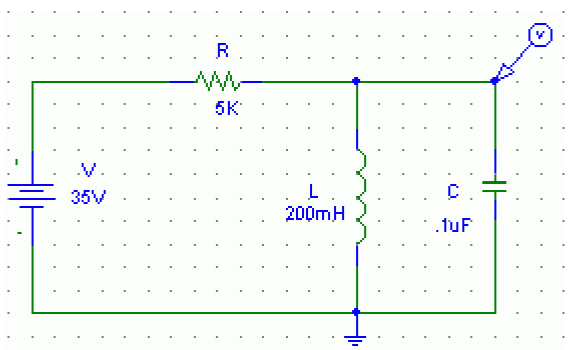
$$V = L\frac{di_o}{dt} + Ri_o + \frac{RLCd^2 i_o}{dt^2}$$

We can transform both sides now.

$$\frac{V}{s} = sLI_o(s) + RI_o(s) + RLCs^2 I_o(s) = I_o(s)\left(sL + R + RLCs^2\right)$$

$$I_o(s) = \frac{V}{s\left(sL + R + RLCs^2\right)} = \frac{V/RLC}{s\left(s^2 + \frac{1}{RC}s + \frac{1}{LC}\right)}$$

Find inverse Laplace of *Io(s)* for the given values in MathCad and Plot the graph of $i_o(t)$

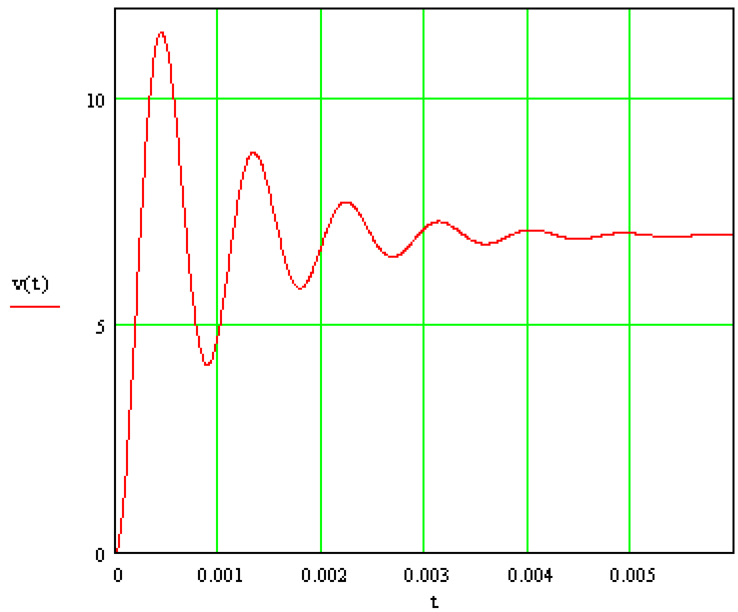


To remember the values We put the circuit here again. If we calculate Io(s) with those values.

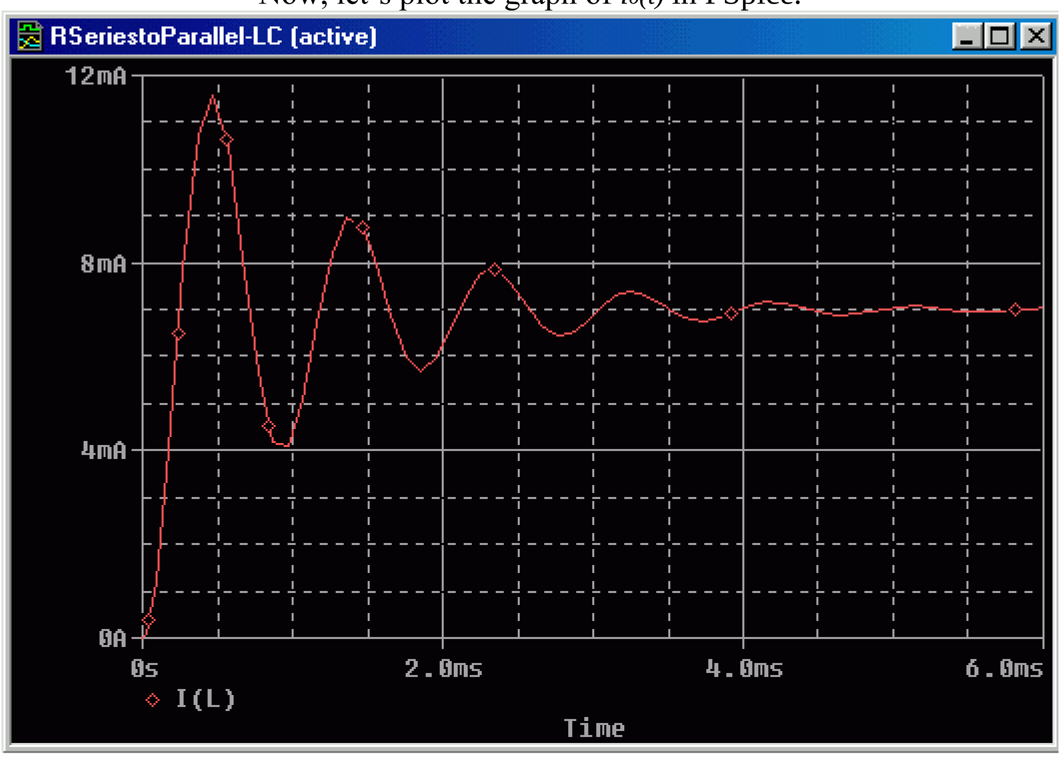
$$i(s) := \frac{350000000}{s \cdot \left(s^2 + 50000000 + 2000 \cdot s\right)}$$

$$7 - 7 \cdot \exp(-1000 \cdot t) \cdot \cos(7000 \cdot t) - \exp(-1000 \cdot t) \cdot \sin(7000 \cdot t)$$

$$v(t) := 7 - 7 \cdot \exp(-1000 \cdot t) \cdot \cos(7000 \cdot t) - \exp(-1000 \cdot t) \cdot \sin(7000 \cdot t)$$



Now, let's plot the graph of $i_o(t)$ in PSpice.



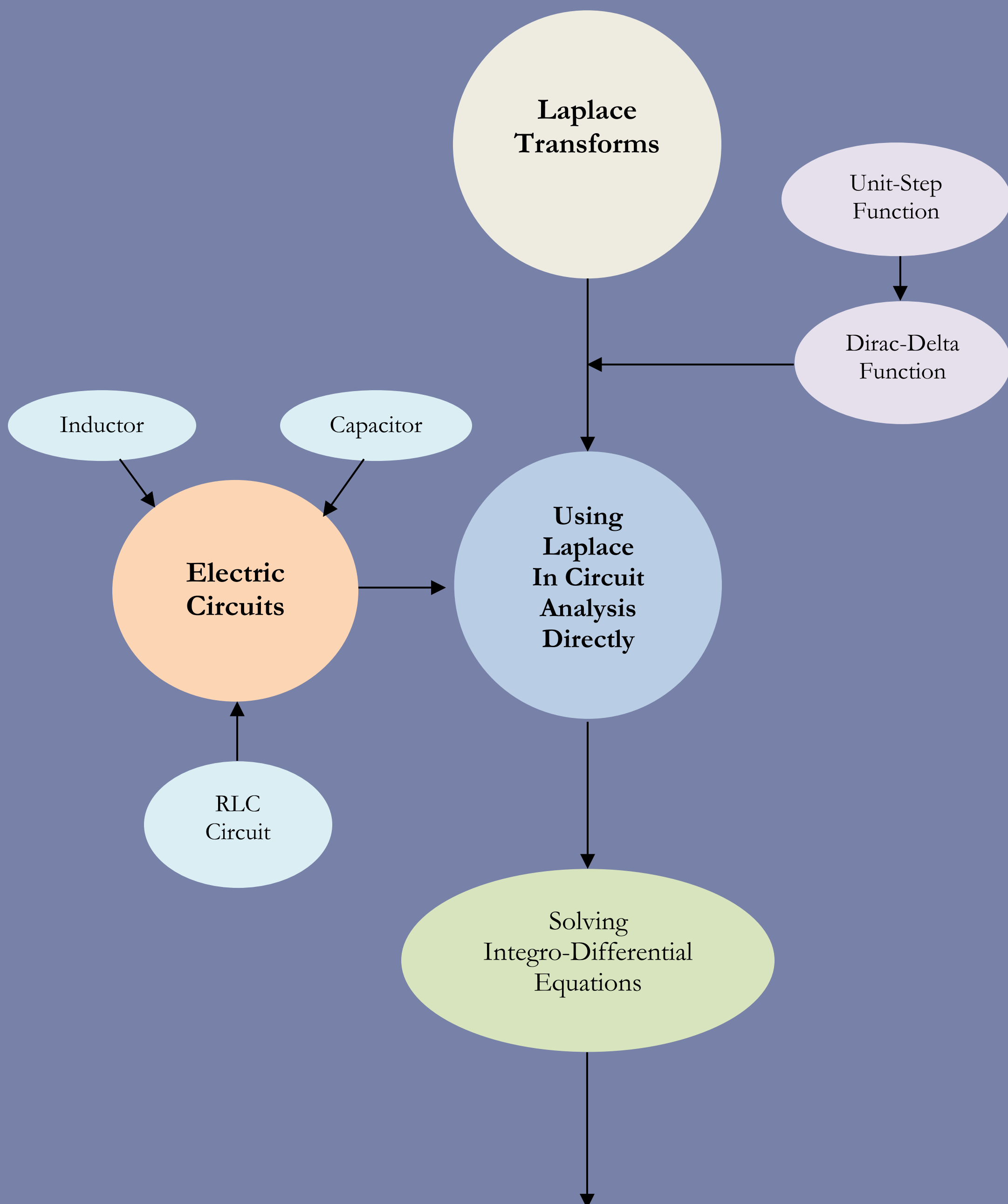
Summary of the Project

So, the basic aim of this project was to learn Laplace Transforms and to get the skill to apply them to electric circuit analysis.

Step by step, we tried to reach this aim. First, we discussed the basic mathematical properties of Laplace transforms. Then, we tried to remember our electric circuit information. And these two different subjects were combined in the second and third section of Laplace Transforms. We tried to show the relation of the two with examples. And in the solution we followed a very specific way.

This specific way can be explained as follows.

- Find the mathematical solution
- See the result of the solution graphically using a computer tool (Like MathCad or Mathematica)
- Draw the circuit in an electric analysis program and find the output experimentally.
- Compare the results.



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